UNIT- IV

Relations and Directed Graphs

Introduction:-

Human language has many words & phrases to describe relationships between or among objects. It may be that for two people A & B, that A is a parent of B A is taller than B, A is infront of B. etc. In algebra it may be that the value of variable x is less than the value of variable y. In set theory it may be that a set 'X' is a subset of a set 'Y' (or) that X is disjoint from Y.

suppose $a < b$, we can say that there is a relation between a & b, we say that 'a is related to b'. The relation is 'less than'. If we form the related elements as an ordered pair i.e (a, b), these types of elements are the elements of a cartesian product A×B.

A×B is read as 'A cross B'.

Cartesian Product:- suppose A & B are any two nonempty sets then the cartesian product of A & B is denoted by 'A×B' & it is defined as

$A \times B = \{(x, y) / x \in A \wedge y \in B\}$.

If the set A consists of 'm' elements & the set B consists of 'n' elements, then we can form 'mn' relation from A to B.

The Relation 'R' is the subset of the Cartesian product $A \times B$.

<u>Eg</u>:- suppose $A = \{1, 2, 3, 4\}$ & $B = \{3, 4, 6, 7, 8\}$.

then $A \times B$ consists of 20 elements.

let $R_1 = \{(1, 3), (3, 6), (2, 6), (3, 8)\}$

$R_2 = \{(1, 4), (1, 6), (2, 4), (3, 6), (3, 7), (4, 7), (4, 8)\}$

$R_3 = \{(1, 3), (1, 4), (2, 3), (2, 7), (3, 8), (4, 8)\}$.

R_1, R_2, R_3 are different relations.

In R_1, 1 is related to 3,

3 " " " 6,

2 " " " 6,

3 " " " 8.

III'ly in R_2 & R_3.

<u>Domain</u> :- The set $\{a \in A / aRb$ for some $b \in B\}$ is called the domain of R.

<u>Range</u> :- The set $\{b \in B / aRb$ for some $a \in A\}$ is called the range of R.

For R_1, the domain = (1, 2, 3) & range = (3, 6, 8)

R_2, " " = (1, 2, 3, 4) & " = (4, 6, 7, 8)

R_3, " " = (1, 2, 3, 4) & " = (3, 4, 7, 8).

Note :- Domain of $R \subseteq A$, Range of $R \subseteq B$.

Eg:- Let $A = \{2, 3, 4\}$ & $B = \{3, 4, 5, 6, 7\}$

Define a relation R from A to B by $(a, b) \in R$ if

a divides b i.e (a/b).

Then $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

Hence domain of R is $\{2, 3, 4\}$ &

range of R is $\{3, 4, 6\}$.

Definition :- Let R be a relation from A to B. Then inverse of relation R from B to A is denoted by R^{-1} & it is defined as $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

Eg:- If $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

is a relation from A to B.

Then $R^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}$

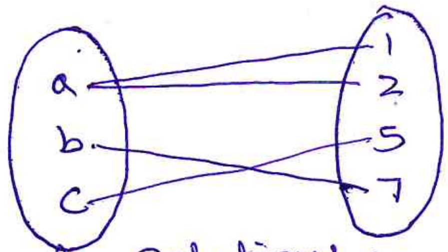
is inverse relation from B to A.

The relations by venn diagram:

Suppose $A = (a, b, c)$ & $B = (1, 2, 5, 7)$.

If there is a relation, then there is a line connecting the element of A to the element of B.

If $R = \{(a,1), (a,2), (b,7), (C,5)\}$.



Properties of Relations:-

A relation 'R' on a set A is said to be

i) Reflexive on A if $(a,a) \in R$ (i.e) $aRa \; \forall \; a \in A$

ii) Symmetric on A if $(a,b) \in R \Rightarrow (b,a) \in R \; \forall \; a, b \in A$

iii) Transitive on A if $(a,b) \in R$, $(b,c) \in R$ $\forall \; a, b, c \in A$.

$\Rightarrow (a,c) \in R$

Note:-

1) If R is reflexive then R' is reflexive

2). The symbol '=' is symmetric. If $a=b$ then $b=a$

3). The symbol '<' is not symmetric.

If $a<b$, then b is not less than a.

A is a son of B is not symmetric

($\because$ B cannot be the son of A).

Eg:-

1) Let $A = \{1, 2, 3, 4\}$ & $R = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,1), (2,1)\}$.

is reflexive & transitive $\because (2,3) \in R, (3,1) \in R \Rightarrow (2,1) \in R$

but not symmetric relation on A.

$\because (3,2) \notin R$ & $(1,3) \notin R$, $(1,2) \notin R$.

2) Let $A = \{1, 2, 3, 4\}$ &

$R = \{(1,3), (4,2), (2,4), (2,3), (3,1)\}$.

R is not symmetric because $(2,3) \in R$ but $(3,2) \notin R$

R is not reflexive

R is not transitive.

$\because (2,3) \in R$, $(3,1) \in R$ but $(2,1) \notin R$.

3) If $a < b$ & $b < c$ then $a < c$, R is transitive.

Anti-symmetric :- A relation R on a set A is called anti-symmetric if α is $(a,b) \in R$ & $(b,a) \in R$ then $a = b$. (or) if aRb, $bRa \Rightarrow a = b \;\forall\, a, b \in A$

A relation R on a set A is said to be

i) irreflexive if $(a,a) \notin R$ i.e a is not related to a. $\forall\, a \in A$

ii) Anti- symmetric on A if $aRb, bRa \Rightarrow a=b$.
$\forall a, b \in A$.

iii) Assymmetric on A if $aRb \Rightarrow b$ is not related to a.
i.e $(a,b) \in R \Rightarrow (b,a) \notin R$ $\forall a, b \in A$.

Give an example of relation which is symmetric but neither reflexive nor anti-symmetric nor transitive.

solution:-
Let $A = \{1,2,3\}$
consider the relation $R = \{(1,1), (1,2), (2,1), (3,2), (2,3)\}$.
on A.

i) R is symmetric.
$\because (1,2) \in R \Rightarrow (2,1) \in R$.
$(3,2) \in R \Rightarrow (2,3) \in R$
$(1,1) \in R \Rightarrow (1,1) \in R$.

Then R is symmetric on A.

ii) R is not reflexive
$\because (2,2) \notin R$, $(3,3) \notin R$.
$\Rightarrow$ R is not reflexive

iii). R is not anti-symmetric
$\because (1,2) \in R$, $(2,1) \in R \Rightarrow 1 \neq 2$.
$\Rightarrow$ R is not anti-symmetric relation on A.

iv) $\cdot$ R is not transitive.

$\because (1,2) \in R,\ (2,3) \in R \Rightarrow (1,3) \notin R.$

$\therefore$ R is not transitive on A.

Give an example of a relation which is transitive but neither reflexive nor symmetric nor antisymmetric

solution:-

Let $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (1,2), (1,3), (2,1), (2,3)\}$ be a relation on A.

$\because (3,3) \notin R$ is not Reflexive.

$\because (1,3) \in R$ but $(3,1) \notin R$ $\therefore$ R is not symmetric.

$\because (1,2) \in R$ & $(2,1) \in R$ but $1 \neq 2$

$\Rightarrow$ R is not anti-symmetric

$\because (1,2) \in R$ & $(2,3) \in R \Rightarrow (1,3) \in R.$

$\Rightarrow$ R is transitive on A.

Give an example of a relation which is anti-symmetric but neither reflexive nor symmetric nor transitive.

solution:-

Let $R = \{(x,y) \in R \times R : x, y$ are integers & $x+3y=12\}$

$\because (1,1) \notin R$ R is not reflexive.

$\because (6,2) \in R$ but $(2,6) \notin R \Rightarrow R$ is not symmetric.

Let $(x,y),(y,x) \in R$.

Then $x+3y = 12 = y+3x$.

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

Hence R is anti-symmetric

& R is not transitive, because $(-6,6), (6,2) \in R$

but $(-6,2) \notin R$.

Let $A = \{1,2,3,4\}$, Relation $\{(1,2),(2,4)\}$ is not reflexive, not symmetric & not transitive on A.

Let $A = \{1,2,3,4\}$ & Relation $R = \{(1,1),(2,2),(3,3),(4,4),(2,3),(3,1),(2,1)\}$ is reflexive & transitive but not symmetric on A.

Let $A = \{1,2,3,4\}$ & relation $R = \{(1,3),(2,3),(2,4),(3,1),(4,2$ then the relation R is irreflexive, assymmetric, not transitive & not anti-symmetric on A.

Equivalence Relations

A relation R on a set A is said to be an equivalence relation on A if

i) R is reflexive on A
ii) R is symmetric on A
iii) R is transitive on A.

Eg:- Let R be a relation on N defined as

$R = \{(a,b) \mid (a,b) \in N \ \forall \ (a+b) \text{ is even}\}$.

then R is an equivalence relation on N.

$\because$ i) $a+a$ is even for any $a \in N$

ii) If $a+b$ is even $\Rightarrow$ $b+a$ is also even $\forall a,b \in N$

iii) If $a+b$ is even, $b+c$ is even

$\Rightarrow a+b+b+c = a+c$ is also even $\forall a,b,c \in N$.

1) show that the relation R is defined $N \times N$ by $(a,b) R (c,d)$ iff $a+d = b+c$ is an equivalence relation.

Solution:-

$N \times N = \{(a,b) \mid a,b \in N\}$.

i) $(a,b) \ R \ (a,b) \ \forall \ (a,b) \in N \times N$

$\because a+d = b+c$.

$\therefore$ R is reflexive on $N \times N$.

ii) Let $(a,b) R (c,d)$ $\forall$ $(a,b), (c,d) \in N \times N$.

$\Rightarrow a+d = b+c$

$\Rightarrow c+b = d+a$.

$\Rightarrow (c,d) R (a,b)$

Hence R is symmetric on $N \times N$.

iii) Let $(a,b) R (c,d)$, $(c,d) R (e,f)$ $\forall$
$\forall (a,b), (c,d), (e,f) \in N \times N$

$\Rightarrow a+d = b+c$ & $c+f = d+e$.

$\Rightarrow a+d+c+f = b+c+d+e$

$\Rightarrow a+f = b+e$

$\Rightarrow (a,b) R (e,f)$

$\Rightarrow R$ is transitive on $N \times N$.

Hence R is an equivalence relation on $N \times N$.

2) Let R be the relation on $N \times N$ which is defined by $(a,b) R (c,d)$ which can be written as $(a,b) R (c,d)$ iff $ad = bc$. P.T R is an equivalence relation.

Solution:-

$N \times N = \{(a,b) \mid a,b \in N\}$.

i) $(a,b)\ R\ (a,b)\ \forall\ (a,b) \in N \times N$.

$\Rightarrow ab = ba$.

$\therefore$ R is reflexive on $N \times N$

ii) Let $(a,b)\ R\ (c,d)\ \forall\ (a,b), (c,d) \in N \times N$.

$\Rightarrow ad = bc$.

$\Rightarrow cb = da$

$\Rightarrow (c,d)\ R\ (a,b)$

$\Rightarrow$ R is symmetric on $N \times N$.

iii) Let $(a,b)\ R\ (c,d)$ & $(c,d)\ R\ (e,f)\ \forall\ (a,b), (c,d), (e,f) \in N \times N$

$\because (a,b)\ R\ (c,d) \Rightarrow ad = bc$

$\because (c,d)\ R\ (e,f) \Rightarrow cf = de$

$\Rightarrow (ad)(cf) = (bc)(de)$

$\Rightarrow a(dc)f = b(cd)e$.

$\Rightarrow a(cd)f = b(cd)e\ [\because dc = cd]$

$\Rightarrow af = be$ by cancelling cd on both sides

$\Rightarrow (a,b)\ R\ (e,f)$

$\Rightarrow$ R is transitive on $N \times N$.

$\because$ it satisfied all 3 properties

$\therefore$ R is equivalence relation on $N \times N$.

3) If R is a relation on the set of integers z defined by $R = \{(x, y) / (x-y)$ is divisible by $3\}$. then prove that R is an equivalence relation.

solution :-

Let us define $R = \{(x,y) / (x-y)$ is divisible by $3\}$

i) For any $x \in X$. $x - x = 0$ is divisible by 3

$\therefore x R x$

$\Rightarrow R$ is reflexive

ii) For any $x, y \in X$,

Let $x R y$ then $x - y$ is divisible by 3.

$\Rightarrow y - x$ is also divisible by 3.

$\Rightarrow y R x$

Hence $x R y \Rightarrow y R x$.

$\Rightarrow$ The relation R is symmetric.

iii) For any $x, y, z \in X$

Let $x R y$ & $y R z$ then $(x-y)$ & $(y-z)$ are divisible by 3

$\Rightarrow (x-y) + (y-z) = x - z$ is also divisible by 3

$\Rightarrow x R z$.

$\Rightarrow$ The relation R is transitive.

$\therefore$ The 3 properties are satisfied.

$\therefore$ The relation R is an equivalence relation.

4) Given $S = \{1, 2, 3, \ldots, 10\}$ & a relation R on S where $R = \{(x,y) / x+y = 10\}$. What are the properties of the relation R?

solution:

Given $R = \{(x,y) / x+y = 10\}$

i.e $R = \{(1,9), (2,8), (3,7), (4,6), (5,5), (6,4),$
$(7,3), (8,2), (9,1)\}$.

i) For any $x \in X$ & $(x,x) \notin R$.

Here, $1 \in X$ but $(1,1) \notin R$.

$\Rightarrow$ the relation R is not reflexive, but it is irreflexive.

ii) $(1,9) \in R \Rightarrow (9,1) \in R$

$(2,8) \in R$, $(8,2) \in R$.

$\vdots$

Hence if $(x,y) \in R$ then $(y,x) \in R$ $\forall (x,y) \in X$.

$\Rightarrow$ the relation R is symmetric, but it is not antisymmetric.

iii) $(1,9) \in R$ & $(9,1) \in R$.

$\Rightarrow (1,1) \notin R$.

Thus, if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z) \notin R$.

$\Rightarrow$ The relation R is not transitive.

Hence R is irreflexive & symmetric.

5) Give an example of a relation that is neither reflexive nor irreflexive.

solution :- Let $X = \{1, 2, 3\}$.

Let us consider the relation $R = \{(1,1), (1,2), (3,2), (2,3), (3,3)\}$.

$\Rightarrow$ The Relation R is neither reflexive nor irreflexive.

[$\because$ $(2,2) \notin R$ $\Rightarrow$ it is not reflexive. &

$\because$ $(1,1) \in R$ & $(3,3) \in R$ $\therefore$ it is irreflexive].

Representation of Relations:

There are two methods of representation of the relation

i) Matrix Method

ii) Directed graph method.

Matrix Method:-

A binary relation (finite relation) R from a set 'A' with 'n' elements to a set 'B' with 'm' elements is represented by 'n×m' matrix called 'Relation matrix' denoted by $M_R = [a_{ij}]$

where $a_{ij} = 1$ if $(a_i, b_j) \in R$.

$= 0$ if $(a_i, b_j \notin R$.

$(a_i, b_j) \in R$ means i^{th} element of A is related to the j^{th} element of B.

eg:- Let $A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$

$R : A \to B$ as $R = \{(a,1), (a,3), (b,2), (b,4), (c,2), (c,3)\}$.

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{3\times 4} \end{array}$$

$\because (a,1) \in R$ [so first row & first column element is 1,

$(a,2) \notin R$ [" " " & 2^{nd} " " 0]

The Relation matrix M_R is also called as the 'Boolean Matrix'.

Eg:- Let $A = \{1, 2, 3, 4, 5, 6\}$, Define Relation R as less than on A then find Relation matrix on A.

Given relation is less than. $R: A \to A$.

where $A = \{1, 2, 3, 4, 5, 6\}$.

then $R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Eg:- Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$ & Relation matrix $M_R = \begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$ then write the relation

$R = \{(a_1, b_2), (a_1, b_4), (a_2, b_1), (a_2, b_3), (a_2, b_4)\}$

Note:- The matrix M_R has the elements as 1 & 0.

Properties:-

i) R is reflexive iff all the elements in the main diagonal of M_R are equal to 1.

ii) R is symmetric if $a_{ij} = a_{ji}$ $\forall$ i, j i.e

$M_R = (M_R)^T$

iii) R is anti-symmetric of $a_{ij} = 1$ with $i \neq j$, then $a_{ij} \cdot a_{ji} = 0$.

In other words either $a_{ij} = 0$ (or) $(a_{ji}) = 0$ when $i \neq j$

suppose the relation R on a set is represented by matrix $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is R reflexive, symmetric, not anti-symmetric

i) $\because$ all the diagonal elements of this matrix are equal to 1, $\therefore$ R is reflexive.

ii) $\because$ $M_R = (M_R)^T$, R is symmetric

iii) R is not anti-symmetric.

) Let $A = \{1, 2, 3, 4\}$ & $B = \{b_1, b_2, b_3\}$. Consider the relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$.

Determine the matrix of the relation.

<u>solution</u> :- Given $A = \{1, 2, 3, 4\}$, $B = \{b_1, b_2, b_3\}$.

$R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$.

(M_R) Matrix of the relation R is written as

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

2) Let $A = \{1, 2, 3, 4\}$. Find the relation R on A determined by the matrix.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

<u>solution</u> :- Given $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$

The relation $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (4, 1), (4, 2), (4, 4)\}$.

Directed Graph Method (Digraph).

Suppose R is relation on set 'A' where $A = \{a_1, a_2 \dots a_n\}$

The elements of A are represented by points (or) circles called nodes (or) vertices of the graph.

An arrow is drawn from the vertex a_i to a_j iff $(a_i, a_j) \in R$. This is called an (directed) edge.

This pictorial representation of 'R' is called a directed graph (or) digraph of R.

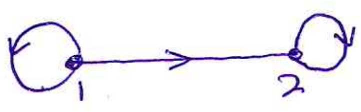
An element of the form (a, a) in a relation corresponds to a directed edge from a to a such an edge is called a loop

(or)

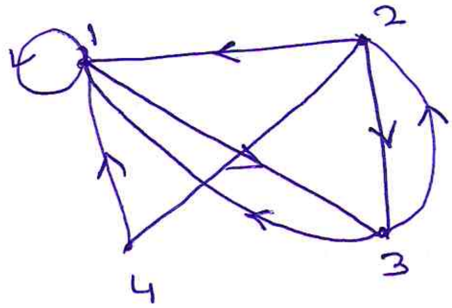
An edge from a vertex to itself is called a loop.

1) Draw the graph for the following relations:

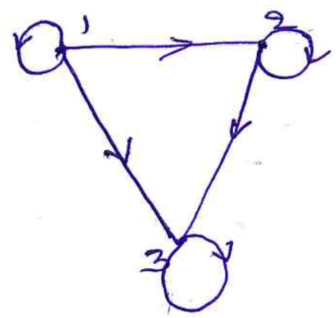
i) $R = \{(1,1), (2,2), (1,2)\}$ on $X = \{1,2\}$



ii) $S = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ on a set $X = \{1,2,3,4\}$.



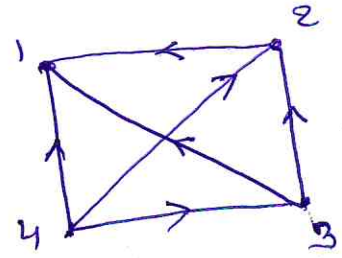
iii) $R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ on
$Y = \{1,2,3\}$



Let $X = \{1,2,3,4\}$ & relation $R = \{(x,y) \mid x > y\}$. Draw the graph of R & also give its matrix.

Given $X = \{1,2,3,4\}$
$R = \{(x,y) \mid x > y\}$.
$= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.

Graph of R.



Matrix of R

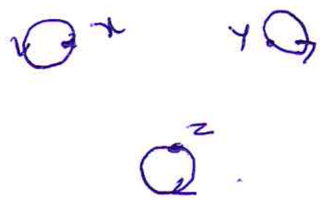
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Properties of relations:

1) If a relation is reflexive, then there is a loop at every point.
2) If a relation is symmetric & if one point is connected to another, then there must be a return arc from the second point to the first.
3) For antisymmetric relations, no direct return paths should exist.
4) If a relation is transitive, then the situation is not so so simple

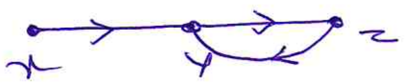
A digraph is <u>reflexive</u> if every vertex has an edge from the vertex to itself (self loop).

$\forall x, y, z$, xRx, yRy, zRz



Reflexive.

A diagraph is <u>irreflexive</u> if none of the vertices have self-loops



$\forall x, y, z$, $x \not R x$.

irreflexive

A diagraph is <u>symmeteric</u> if for every edge in one direction b/w points there is also an edge in the opposite direction between the same two points.

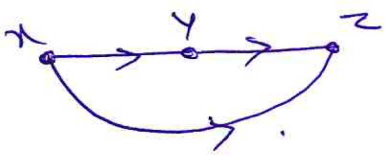
symmeteric.

A diagraph is <u>Anti-symmeteric</u> if no two distinct points have an edge going between them in both directions.

$\forall a, b \in A$
if $(a, b \in R) \Rightarrow (b, a) \in R$
unless $a = b$.

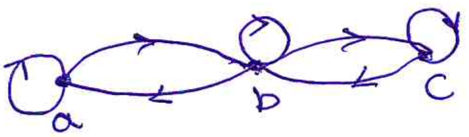
If R is an anti-symmeteric relation, then for different vertices i & j there cannot be edge from vertex i to vertex j & an edge from vertex j to vertex i.

A digraph is transitive if for any 3 vertices x, y & z whenever there is an edge from x to y & an edge from y to z there is also an edge from x to z.

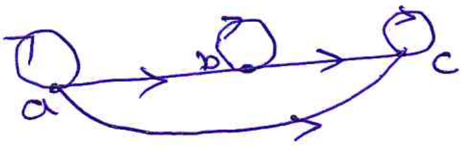


Give an example of a non-empty set & a relation on the set that satisfies each of the following combinations of properties, draw a digraph of the relation.

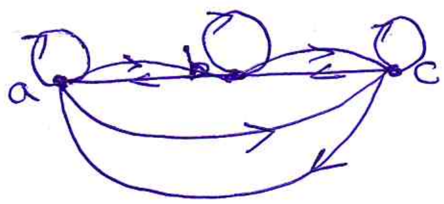
1) symmetric & reflexive but not transitive.



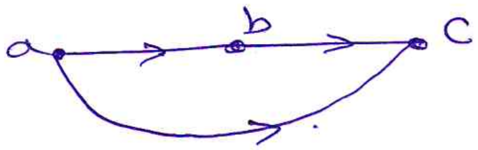
2) Transitive & reflexive but not symmetric.



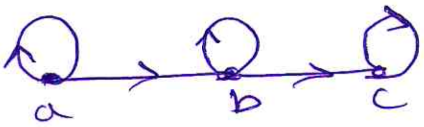
3) Transitive & reflexive, symmetric but not anti-symmetric.



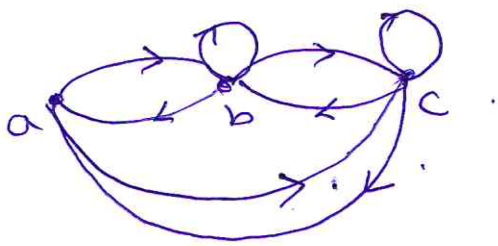
4) Transitive & anti-symmetric but not reflexive



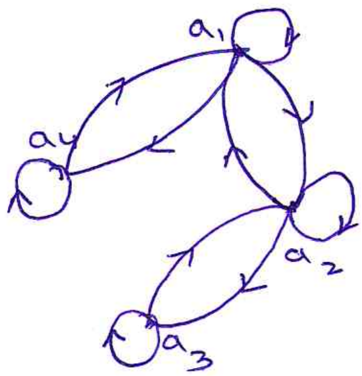
5) Anti-symmetric & reflexive but not transitive.



6) symmetric, transitive but not reflexive.



Find the relation determined by the given graph & the corresponding relation matrices. Also, determine the properties of the relation given by the graphs.



Solution :-

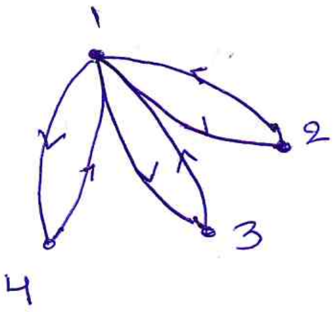
$R = \{(a_1,a_1), (a_1,a_2), (a_2,a_1), (a_2,a_2), (a_2,a_3), (a_3,a_2),$
$(a_3,a_3), (a_1,a_4), (a_4,a_1), (a_4,a_4)\}$.

The corresponding matrix of the relation is written as

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The relation is reflexive [∵ every vertex has a loop) & symmetric (∵ whenever $a_i R a_j$ then $a_j R a_i$). i.e $M_R = (M_R)^T$.

The relation is not transitive.

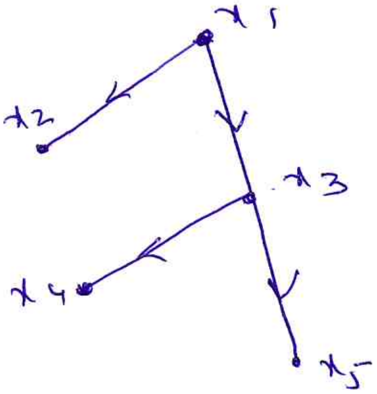


$R = \{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation is not reflexive $\because$ loops are not present in graph. It is symmetric i.e $M_R = (M_R)^T$.

It is not transitive, not anti-symmetric.



$R = \{(x_1, x_2), (x_1, x_3), (x_3, x_4), (x_3, x_5)\}$.

$$M_R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation is not reflexive $\because$ loops are not present in graph.

It is not symmetric $\because M_R \neq (M_R)^T$.

It is not transitive

The relation is anti-symmetric $\because$ no direct return path exist.

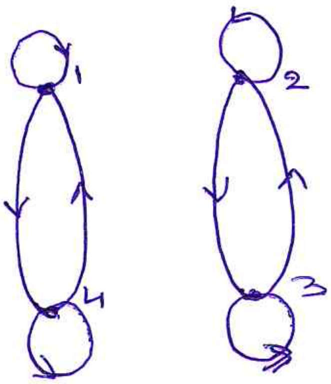
Let $X = \{1, 2, 3, 4\}$ & $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$.

P.T R is an equivalence relation.

The Matrix of R will be written as.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding graph of R.



1) The Relation is Reflexive
∵ every vertex has a loop.

2) It is symmetric.
∵ $M_R = (M_R)^T$

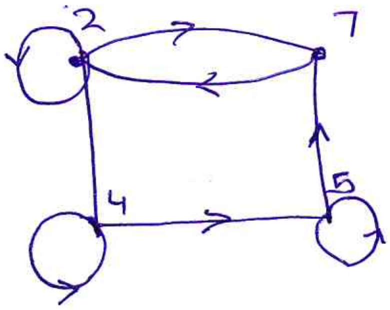
3). It is transitive
∵ $(1,4) \in R$, $(4,1) \in R \Rightarrow (1,1) \in R$

Hence it satisfied all the 3 properties ∴ R is an equivalence relation.

Let $A = \{2, 4, 5, 7\}$ & Let R be the Relation on A having the matrix. $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

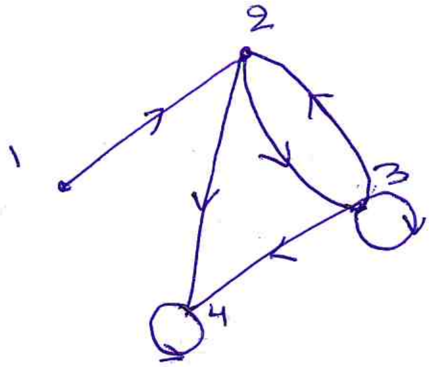
Construct the digraph of R.

$R = \{(2,2), (2,4), (2,7), (4,4), (4,5), (5,5), (5,7), (7,2)\}$



Find the relation R determined by each of the digraphs given below. Also write the matrix of relation.

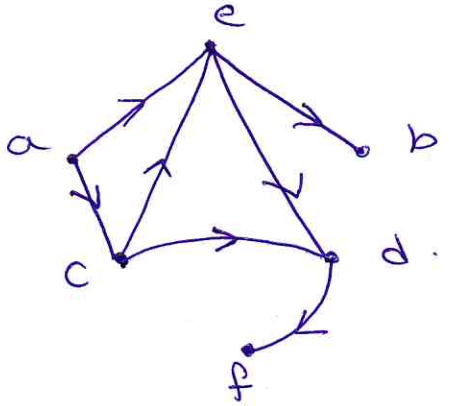
i)



$R = \{(1,2), (2,3), (3,4), (4,4), (2,4), (3,2), (3,3)\}$.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii).



$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$R = \{(a,c), (a,e), (c,e), (c,d), (e,d), (e,b), (d,f)\}$.

consider the relation $R = \{(1,3), (1,4), (3,2), (3,3), (3,4)\}$ on $A = \{1,2,3,4\}$.

1) Find the matrix representation M_R of R.

2) Find R^{-1}

3) Draw the directed graph of R.

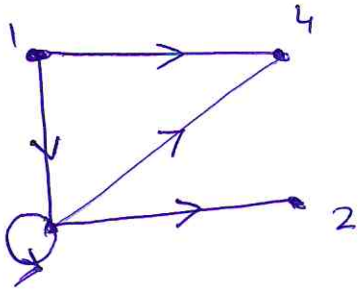
1)

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2) $R^{-1} = \{(3,1), (4,1), (2,3), (3,3), (4,3)\}$.

3)



Partial ordering Relations:

A relation R on a set P is called a partial order relation or a partial ordering in P iff R is reflexive, antisymmetric & transitive. We denote a partial ordering by the symbol $\leq$

A set P on which a partial ordering $\leq$ is defined is called a partially ordered set (or) a poset & it is denoted by $(P, \leq)$ or $[P, \leq]$

The characteristic properties of a partial order can be described as follows.

1) $\forall a \in A$, $a \leq a$ (reflexive).
2) $\forall a, b \in A$ if $a \leq b$ & $b \leq a$ then $a = b$ (Antisymmetry)
3) $\forall a, b, c \in A$ if $a \leq b$, $b \leq c$ then $a \leq c$ (Transitive).

eg:- $[Z, <]$ is not a poset because $<$ is not reflexive. $[Z, \leq]$ is a poset.

eg:- Let A be any set & $P(A)$ be the collection of all subsets of A. Then $[P(A), \subseteq]$ is a poset.

$A = [a, b]$ $\quad$ $P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Let $[P, \leq]$ be a Poset. Elements a, b in P are said to be comparable. if either $a \leq b$ or $b \leq a$ otherwise they are incomparable.

Let $(P, \leq)$ be a Poset. If every pair of elements of A are comparable, then $(P, \leq)$ is called a totally ordered set (or) chain (or) simply ordered set. Here the relation $\leq$ is called a total order (or) linear order (or) a simple order on P.

HASSE DIAGRAM (OR) POSET DIAGRAM.

A Partial ordering $\leq$ on a set P can be represented by means of a diagram known as HASSE diagram (or) Poset diagram of $(P, \leq)$.

The Procedure for drawing Hasse diagram for a. poset P as follows:.

1) Each element is represented by a small circle or a dot.

2) The circle for $a \in P$ is drawn below the circle for $b \in P$ if $a < b$.

3) A line is drawn between a & b if b covers a i.e $a < b$.
if b does not cover a, then a & b are not connected directly by a single line.

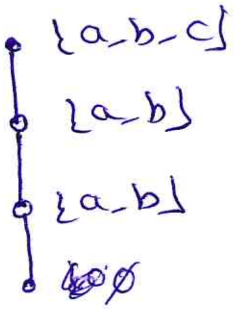
For a totally ordered set $(P, \leq)$ the Hasse diagram consists of circles one below the other. Thus poset is called a chain.

eg:- Let $P_1 = [\emptyset, \{a\}, \{b\}, \{a, b, c\}]$

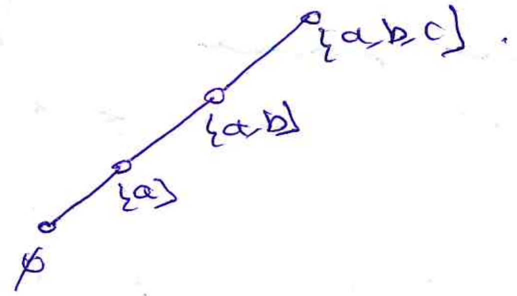
$P_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}]$.

P_1 is not a totally ordered set but P_2 is a Poset

$\therefore \emptyset \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$.



(or)



Note:- The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, & a line is drawn between x & y if y cover x.

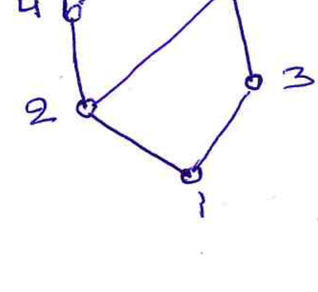
If $x < y$ but y does not cover x, then x & y are not connected directly by a single line.

1) Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Let $P = \{1, 2, 3, 4, 6, 8, 12\}$

$(P, \leq) = \{(1, 2), (1, 3), (1, 4), (1, 6), (1, 8), (1, 12), (2, 4), (2, 6), (2, 8), (2, 12), (3, 6), (3, 12), (4, 8), (4, 12), (6, 12)\}$.

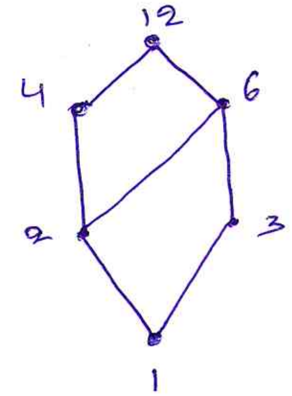
Hasse diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, /)$ is



2) Draw Hasse diagram of Poset $(D_{12}, /)$.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$\{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12),$
$(3,6), (3,12), (4,12), (6,12)\}$.



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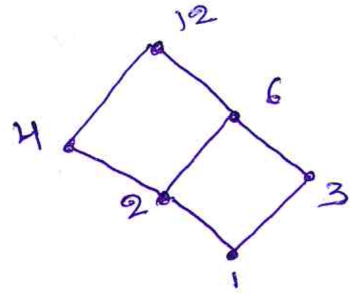
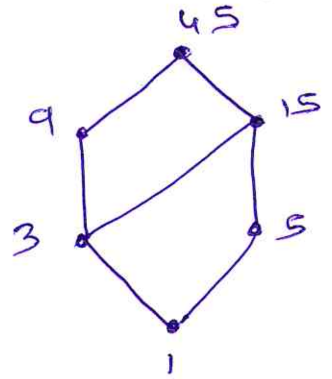


3) Draw the Hasse diagram of Poset $(D_{45}, /)$.

$D_{45} = \{1, 3, 5, 9, 15, 45\}$



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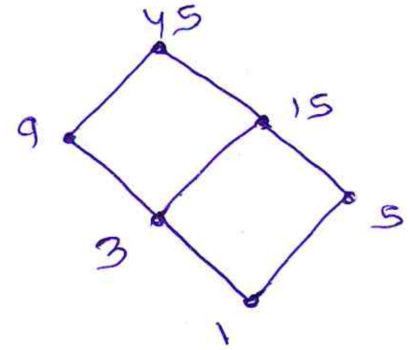
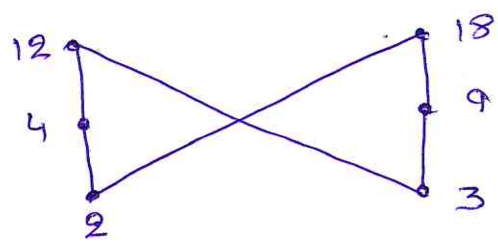
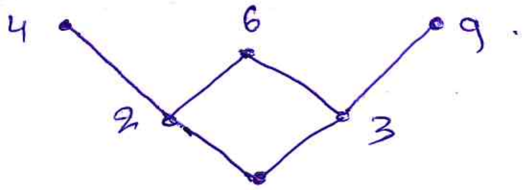


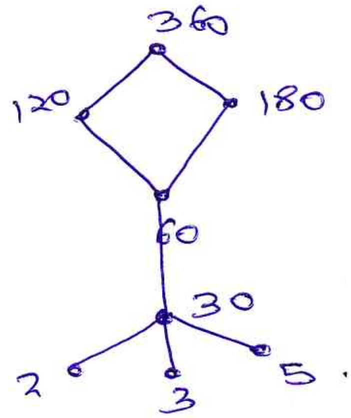
4) Draw the Hasse diagram of $[\{2, 3, 4, 9, 12, 18\}, /]$



5) Draw Hasse diagram of $[\{1,2,3,4,6,9\}; /]$



6) Draw Hasse diagram of $[\{2,3,5,30,60,120,180,360\}; /]$

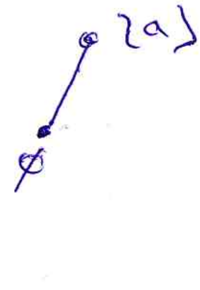


7) Let A be any finite set & P(A) be the power set of A, $\subseteq$ be the inclusion relation on the elements of P(A). Draw Hasse diagram of $(P(A), \subseteq)$ for

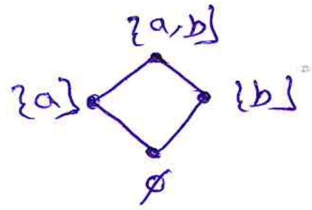
i) $A = \{a\}$

$P(A) = \{\emptyset, \{a\}\}$.

$\therefore [P(A), \subseteq]$ is a Poset



ii) $A = \{a, b\}$

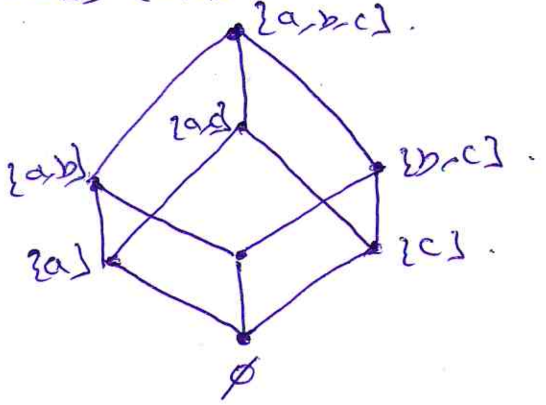


$P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

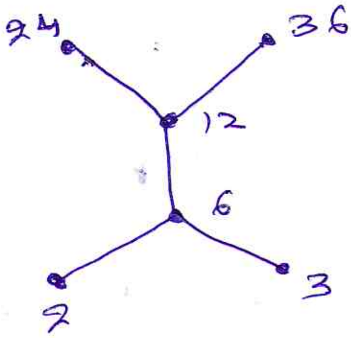
$\therefore$ $\boxed{P(A), \subseteq}$ is a Poset.

iii) $A = \{a, b, c\}$

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$.



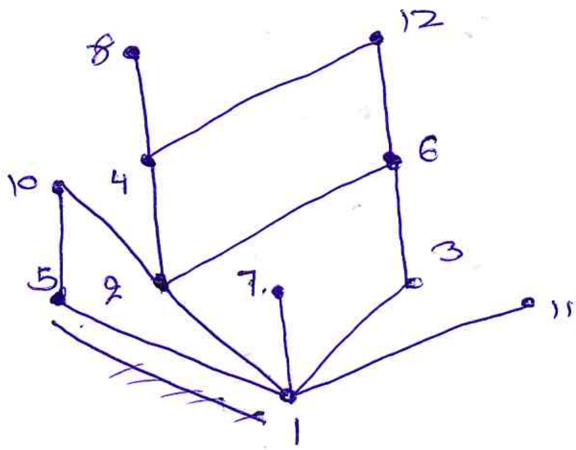
8) If $X = \{2, 3, 6, 12, 24, 36\}$. & the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $\{X, \leq\}$



Note :- on a poset diagram there is a vertex for each element of A, all loops are omitted, eliminating explicit representation of the reflexive property. An edge is not present in a poset diagram if it is implied by the transitivity of the relation. If we write $x < y$ to mean $x \leq y$ but $x \neq y$, then an edge connects a vertex x to a vertex y iff y covers x, i.e iff there is no other element $z \ni x < z$ & $z < y$.

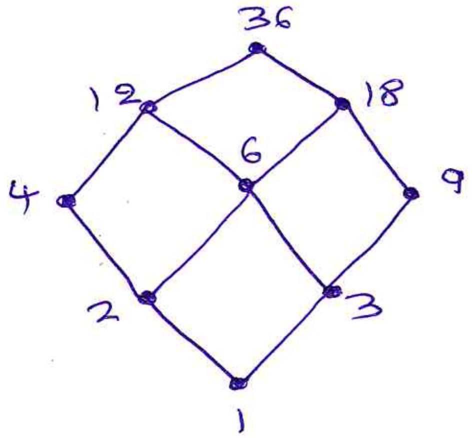
9) Let I_{12} is the set of all positive integers which are less than or equal to 12. Draw Hasse diagram of $(I_{12}, /)$.

solution :- $I_{12} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.



10) Draw the Hasse diagram representing the positive divisors of 36.

{1, 2, 3, 4, 6, 9, 12, 18, 36}.



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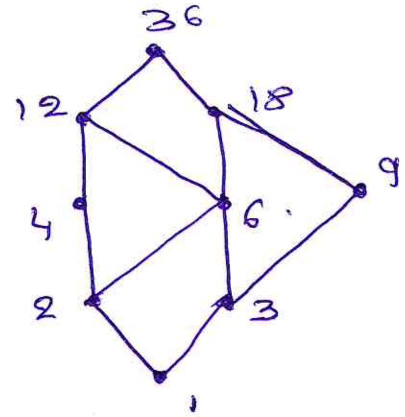


11) Let X = {2, 3, 6, 12, 24, 36} then prove (X, /) is a poset & draw its Hasse diagram.

Let $a \in X$, $a/a \Rightarrow$ / is reflexive on X

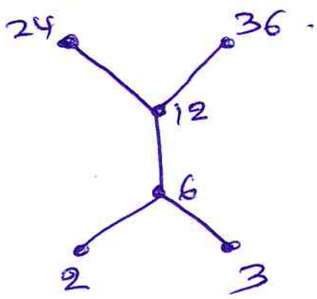
Let $a, b \in X$, a/b, $b/a \Rightarrow a = b$

$\Rightarrow$ / is antisymmetric on X.

Let $a, b, c \in X$, a/b, $b/c \Rightarrow a/c$

$\Rightarrow$ / is transitive on X.

Its Hasse diagram is



Let $(P, \leq)$ be a Poset, $\forall$ $A \leq P$, then $a \in A$ is called a lower bound of A if $a \leq x$ $\forall$ $x \in A$ & if there are no lower bounds of A which are greater than a, then a is called greatest lower bound (g.l.b) of A (or) infimum (inf) of A.

Let $(P, \leq)$ be a Poset, $A \leq P$, then $a \in A$ is called an upper bound of A if $x \leq a$ $\forall$ $x \in A$ and if there are no upper bounds of A which are less than a then a is called least upper bound (l.u.b) of A (or) supremum (sup) of A.

An element $a \in A$ is called an upper bound of a subset B of A if $x R a$ $\forall x \in B$.

An element $a \in A$ is called a lower bound of a subset B of A if $a R x$ $\forall$ $x \in B$.

An element $a \in A$ is called a least upper bound (LUB) of a subset B of A if the following 2 conditions hold:

i) a is an upper bound of B.

ii) If a is an upper bound of B then $a R a'$

A least upper bound is also called as supremum, written as 'sup'.

An element $a \in A$ is called a greatest lower bound (GLB) of a subset B of A if the following 2 conditions hold.

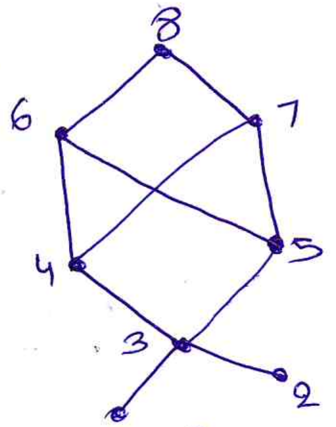
i) a is a lower bound of B.

ii) If a' is a lower bound of B then $a' R a$.

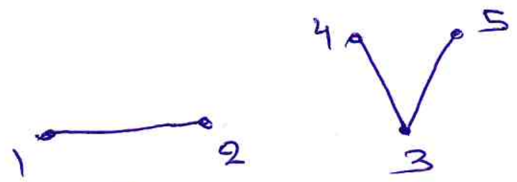
A greatest lower bound is also called an infimum written as "Inf".

Note :- Every poset has atmost one greatest element & atmost one least element.

Eg:-



B



B_1

$1 R 3$, $2 R 3$, ∴ 3 is an upper bound of B_1, |||^{ly} 4, 5, 6, 7, 8 are also upper bounds of B_1.

It has no greatest lower bound, it has lub is 8.

Special elements in Posets (or) External of elements in Posets.

An element $a \in A$ is called a <u>maximal element</u> of A if whenever there is $x \in A \ni aRx$ then $x = a$.

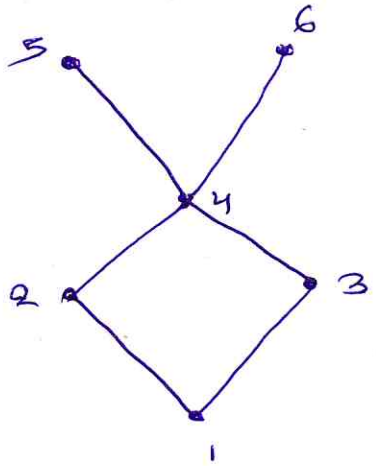
This means that a is a maximal element of A iff in the Hasse diagram of R no edge starts at a.

An element $a \in A$ is called a <u>minimal element</u> of A if whenever there is $x \in A \ni xRa$ then $x = a$.

This means that a is a minimal element of A iff in the Hasse diagram of R no edge terminates at a.

An element $a \in A$ is called a <u>greatest element</u> of A if $xRa \; \forall \; x \in A$.

An element $a \in A$ is called a <u>least element</u> of A if $aRx \; \forall \; x \in A$.

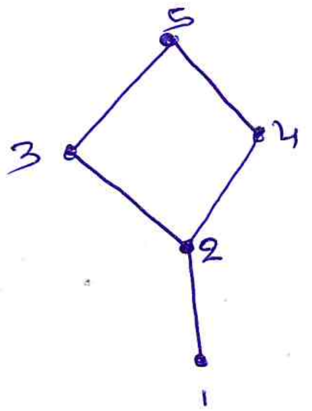


5 & 6 are maximal elements

1 is a minimal element

1 is the least element

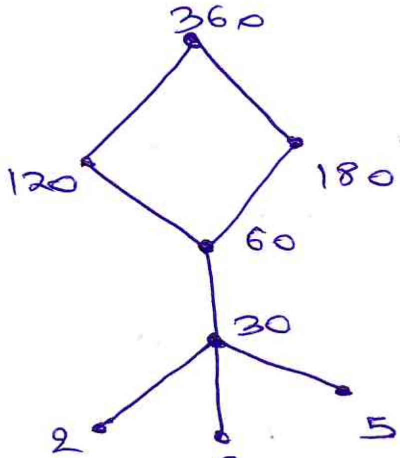
There is no greatest element.



5 is a maximal as well as a greatest element.

1 is a minimal as well as least element.

1) Consider the Poset {(2, 3, 5, 30, 60, 120, 180, 360), /} having Hasse diagram shown below.



g.l.b of {120, 180, 360} is 60

The set {2, 3, 5, 30} has no lower bounds & no g.l.b.

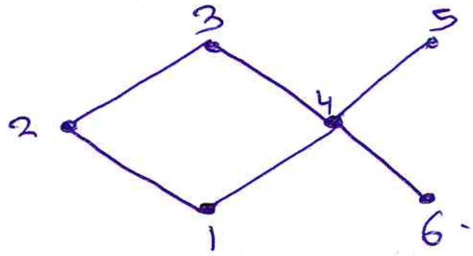
The set {60, 120, 180, 360} of the poset has a minimal & least element is 60.

The set {120, 180, 360} has a minimal elements - 120 & 180 but no least element.

The set {120, 180, 360} has a g.l.b is 60 but the set {2, 3, 5, 30} has no lower bounds & hence no g.l.b.

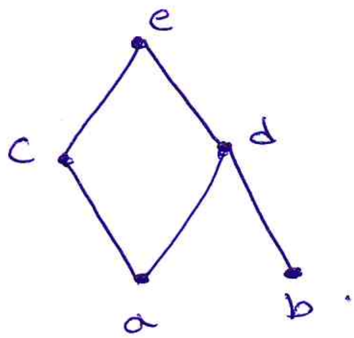
The set {60, 120, 180} has a l.u.b is 360.

2) For the posets shown in the following Hasse diagrams, find all maximal elements & all minimal elements:



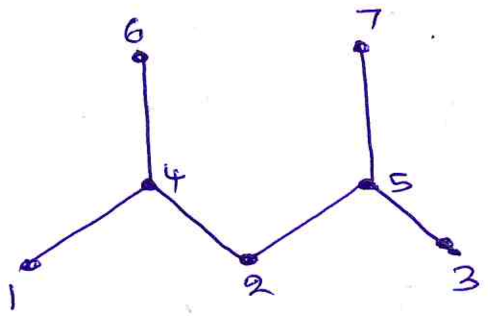
Minimal elements - 1, 6

Maximal elements - 3, 5



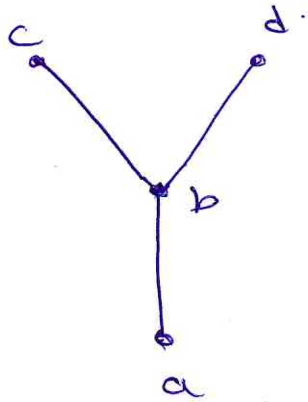
Maximal element - e

Minimal element - a, b.



Maximal elements - 6, 7

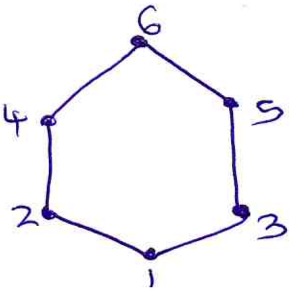
Minimal elements - 1, 2, 3.



Maximal elements - c, d

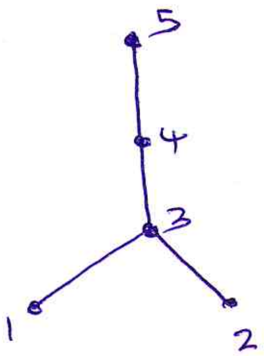
Minimal element - a.

3) For the posets shown in the following Hasse diagrams, find the greatest & least elements.



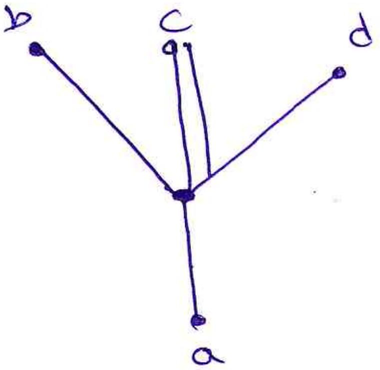
Greatest element - 6

least element - 1



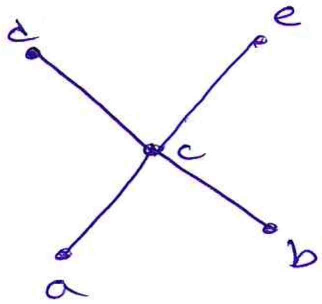
Greatest element - 5

least element - None.

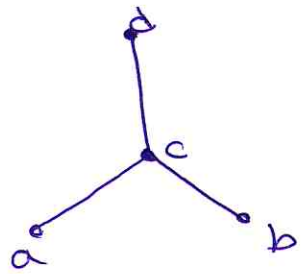


Greatest element - None

least element - a.

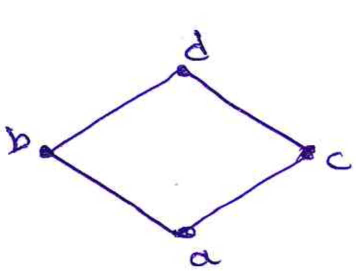


The poset has neither a least element nor a greatest element.



Greatest element - d.

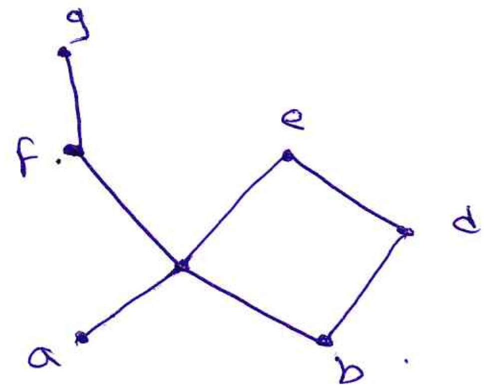
least element - None.



Greatest element - d

least element - a.

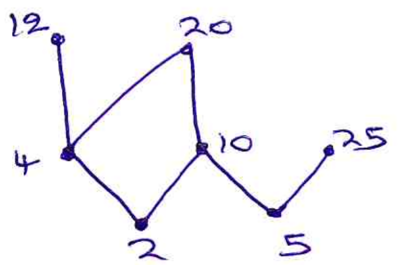
4) Find the maximal & minimal elements of the Poset A whose Hasse diagram is given below:



Maximal element - g, e.

Minimal element - a, b.

which elements of the posets {(2, 4, 5, 10, 12, 20, 25), /} are maximal & minimal.



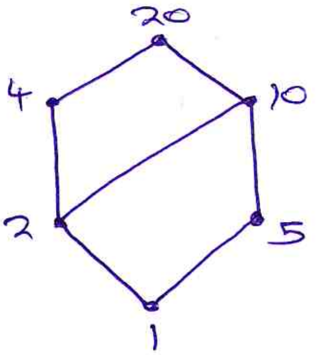
Maximal element - 12, 20, 25.

Minimal element - 2, 5

5) Draw a Poset diagram for each of the following Posets & determine all maximal, minimal elements, greatest & least elements if they exist.

$(D_{20}, /)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$.



Maximal element - 20

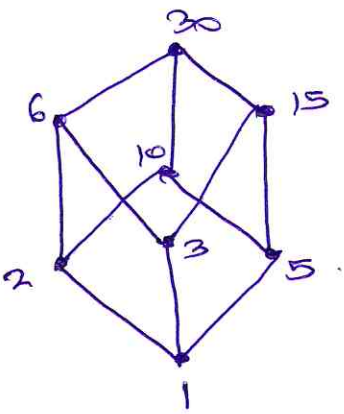
Minimal element - 1

Greatest element - 20.

least element - 1

$(D_{30}, /)$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



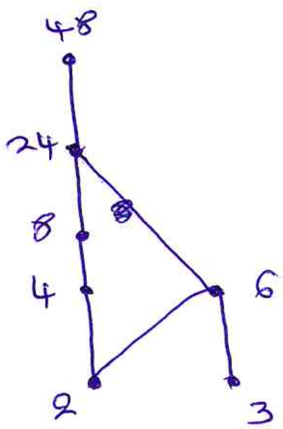
Maximal - 30

Minimal - 1

Greatest - 30

least - 1

$(A, /)$ where $A = \{2, 3, 4, 6, 8, 24, 48\}$.



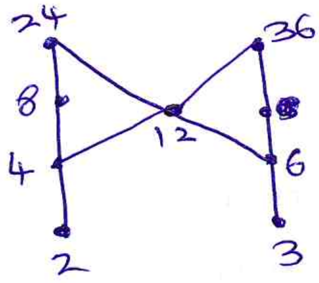
Maximal - 48

Minimal - 2, 3.

Greatest - 48

least - does not exist.

(A, /) where A = {2, 3, 4, 6, 12, 18, 24, 36}



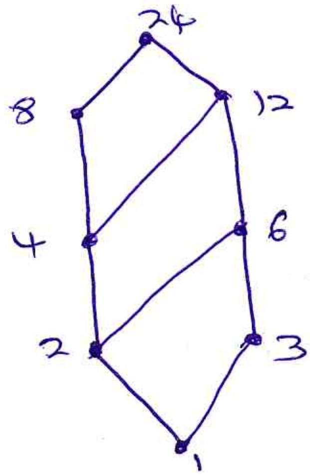
Maximal – 24, 36.

Minimal – 2, 3.

Greatest – does not exist

least – does not exist.

6) Let D_{24} = {1, 2, 3, 4, 6, 8, 12, 24} & relation divides '/' be a partial ordering on D_{24}. Then draw the Hasse diagram of (D_{24}, /) & also find the following :



i) all lower bounds of 8, 12 – 1, 2, 4

ii) all upper bounds of 8, 12 – 24

iii) g.l.b of 8, 12 – 1

iv) l.u.b of 8, 12 – 24

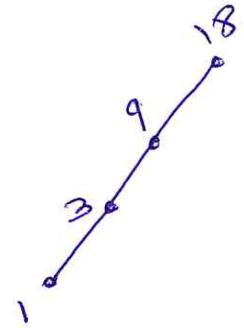
v) greatest & least elements of this poset if exists.

greatest – 24

least – 1

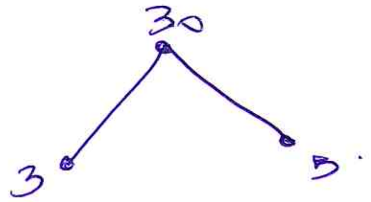
7) Draw the Hasse diagrams of the following sets under the partial ordering relation 'divisibility'.

i) $\{1, 3, 9, 18\}$.



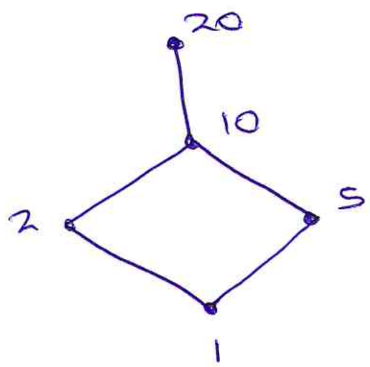
This is a totally ordered set in which the least element is 1 & the greatest element is 18.

ii) $\{3, 5, 30\}$.



There exists no least element in the Hasse diagram.

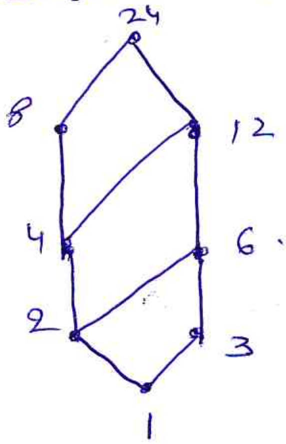
iii) $\{1, 2, 5, 10, 20\}$.



least element is 1
greatest element is 20.

8) Draw the Hasse diagram representing the Positive integers of 24. Find Minimal, Maximal, Greatest & least elements.

Hasse diagram of $(D_{24}, /)$



24 is a maximal as well as greatest element.

1 is a minimal as well as least element.

Lattice :-

A lattice is a Poset in which each Pair of elements has aleast upper bound (l.u.b) & greatest lower bound (g-l-b).

(or)

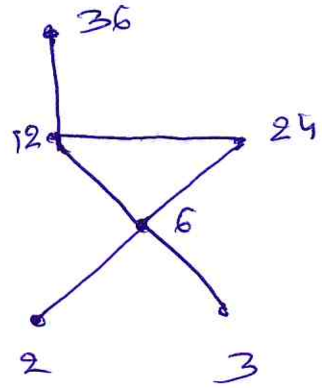
A lattice is both a Join - semi lattice & a meet - semilattice.

we define a join-semi lattice as a Poset [A, ≤] in which each pair of elements a & b of A have a l.u.b. we call this l.u.b the join of a & b. It is denoted by $a \vee b$ (or) sum of a, b i.e $a \oplus b$.

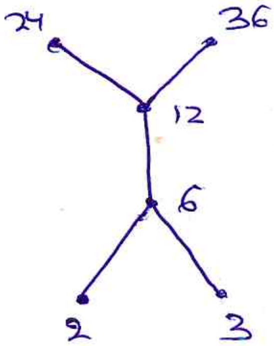
we define a meet-semilattice as a Poset in which each Pair of elements a & b have a g.l.b this g.l.b is called the meet of a & b, it is denoted by $a \wedge b$ (or) Product of a, b i.e $(a \times b)$

Note:- every lattice is a poset but all Posets are not lattices.

Eg:- $X = \{2, 3, 6, 12, 24, 36, 1\}$ is a Poset but not lattice because g.l.b of $(2, 3)$ is $1 \notin X$.



(or)



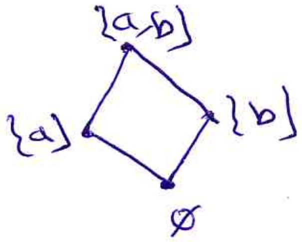
Note :-

g.l.b $\{a, b\} = a \wedge b$. called the meet of a & b.

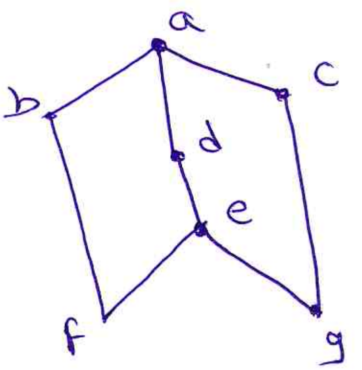
l.u.b $\{a, b\} = a \vee b$ called the join of a & b.

Eg:- let $S = \{a, b\}$, $P(S) = [\emptyset, \{a\}, \{b\}, \{a, b\}]$.

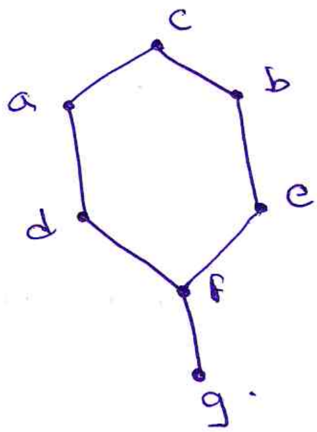
Then $[P(S), \subseteq]$ is a lattice because g.l.b & l.u.b of any pair of elements of $P(S)$ are belongs to $P(S)$.



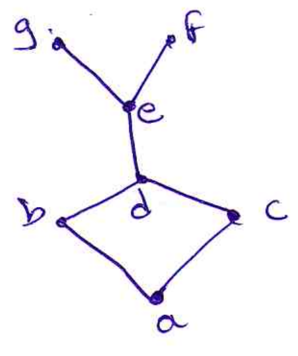
1) which of the following posets are lattices on the set {a, b, c, d, e, f, g}.



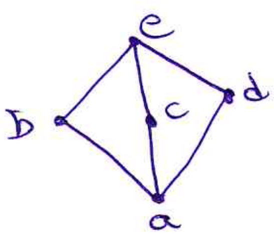
Poset is not a lattice because there is no lower bound for {f, g}.



Poset is a lattice because for every pair of elements have l.u.b & g.l.b in Poset.

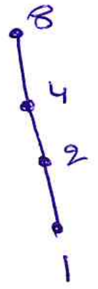


Poset is not a lattice because l.u.b does not exist.

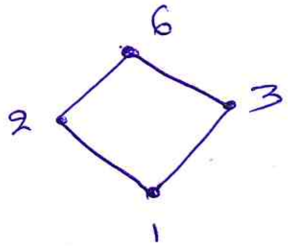


Poset is a lattice, it has g.l.b & l.u.b.

$[D_8 ; /]$ is a lattice



$[D_6 ; /]$ is a lattice



g.l.b of $(2,3)$ or $2 \wedge 3$ is 1

l.u.b of $(2,3)$ or $2 \vee 3$ is 6.

Note :-
Any totally ordered set is a lattice in which $a \vee b$ is simply the greater & $a \wedge b$ is the lesser of a & b.

$$a \vee b = \max\{a, b\}$$

$$a \wedge b = \min\{a, b\}.$$

The Poset $[P, /]$ where P is the set of Positive integers, is a lattice in which

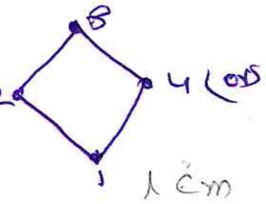
$$a \wedge b = \gcd(a, b)$$

$$a \vee b = \text{lcm}(a, b).$$

Eg :- $6 \wedge 9 = 3$, $6 \vee 9 = 18$

2) Define a lattice ? Show that $(D_8, /)$ is a lattice, where D_8 is the set of all divisors of 8.

$D_8 = \{1, 2, 4, 8\}$.



$2 \underline{|2, 4}$
$\quad 1, 2$

$2 \underline{|2, 8}$
$\quad 1, 4$

gcd

g.l.b	1	2	4	8
1	1	1	1	1
2	1	2	2	2
4	1	2	4	4
8	1	2	4	8

lcm

l.u.b	1	2	4	8
1	1	2	4	8
2	2	2	4	8
4	4	4	4	8
8	8	8	8	8

It is a lattice ∵ each pair of elements has a least upper bound (l.u.b) & a greatest lower bound (g.l.b).

Show that $(D_6, /)$ is a lattice.

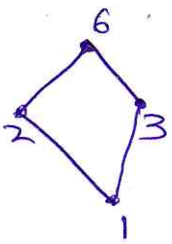
$D_6 = \{1, 2, 3, 6\}$. It is a lattice because every pair of elements of D_6 have g.l.b & l.u.b in D_6.

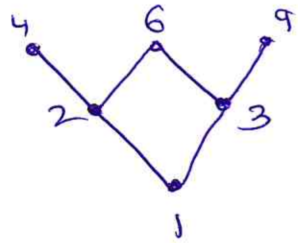
$a \wedge b$ (gcd)

g.l.b	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

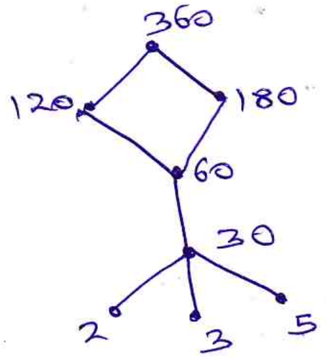
$a \vee b$ (lcm)

l.u.b	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

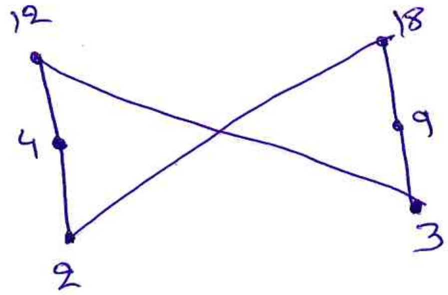




It is the meet semilattice i.e it has g.l.b & no l.u.b.
i.e g.l.b is 1 ; 4 & 6 have no l.u.b.



It is a join semilattice
i.e it has l.u.b & no g.l.b.
∵ 2 & 3 have no g.l.b.
i.e l.u.b is 360.



It has neither g.l.b nor l.u.b.
It is not a lattice.

Let $(L, \leq)$ be a lattice in which $\wedge$ & $\vee$ denote the operations of meet & join respectively for any $a, b, c \in L$.

i) $a \wedge a = a$; $a \vee a = a$ (idempotent)

ii) $a \wedge b = b \wedge a$; $a \vee b = b \vee a$ (commutative).

iii) $\left.\begin{array}{l}(a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c)\end{array}\right\}$ associative.

iv) $\left.\begin{array}{l}a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a\end{array}\right\}$ absorption.

Operations on Relations.

Let R & S be relations from a set A to set B. Then,

i) $R \cup S = \{(a,b) / (a,b) \in R$ or $(a,b) \in S\}$

ii) $R \cap S = \{(a,b) / (a,b) \in R$ & $(a,b) \in S\}$

iii) $R - S = \{(a,b) \in R - S, (a,b) \in R, (a,b) \notin S\}$

iv) R' or $R^c = \{(a,b) \in R', (a,b) \notin R\}$.

Eg:- let $R = \{(1,3), (1,5)\}$ & $S = \{(2,4), (1,5)\}$.

$R \cup S = \{(1,3), (1,5), (2,4)\}$

$R \cap S = \{(1,5)\}$.

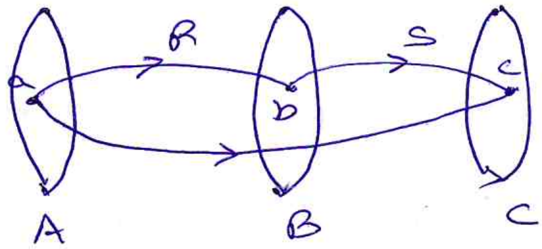
Inverse of R :- let R be a relation from a set A to B. The inverse of R is relation from B to A & is given by

$\cancel{R^{-1}} = \bar{R}' = \{(y,x) / (x,y) \in R\}$.

Let R be a relation from A to B & S be a relation from B to C. Then composite relation of R & S is denoted by $R \circ S$ (or) RS & it is defined as

$R \circ S = \{(a,c) \,/\, a \in A,\ c \in C\ \exists\ b \in B \ni (a,b) \in R,$
$(b,c) \in S\}.$

Composition of Relations.



<u>Note:-</u>

i) $R \subseteq A \times B$ (ii) $S \subseteq B \times C$ (iii) $R \circ S \subseteq A \times C$

iv) $S \circ R \neq R \circ S$.

Let R & S be two relations on

$A = \{1,2\}$ & $R = \{(1,1),(2,2)\}$ $S = \{(1,2),(2,1)\}$.

Then $R \circ S = \{(1,2),(2,1)\}$.

1) Let $A = \{a,b,c\}$, $B = \{1,2,3\}$, $C = \{x,y,z\}$.

$R = \{(a,1),(a,3),(c,2),(a,2),(b,3)\}$.

$S = \{(1,y),(1,z),(2,x),(2,z),(3,x),(3,y)\}$.

$R \circ S = \{(a,y),(a,x),(c,z),(a,z),(b,x),$
$(b,y),(c,x)\}$.

$S \circ R = \{\}$ or $\emptyset$

$\therefore R \circ S \neq S \circ R$.

2) Let R, S be two relations on $A = \{1, 2, 3\}$ &

$R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}$

$S = \{(1,2), (1,3), (2,1), (3,3)\}$; then find the following:

$R \cup S = \{(1,2), (1,1), (2,3), (3,3), (3,1), (1,3),$
$(2,1)\}$

$R \cap S = \{(1,2), (3,3)\}$.

$R - S = \{(1,1), (2,3), (3,1)\}$.

$R' = (A \times A) - R$.

$A \times A = \{1,2,3\} \times \{1,2,3\}$

$= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3),$
$(3,1), (3,2), (3,3)\}$.

$R' = \{(1,3), (2,1), (3,2), (2,2)\}$.

$R \circ S = \{(1,2), (1,1), (2,3), (3,2), (3,3), (1,3)\}$

$S^2 = S \circ S$.

$S = \{(1,2), (1,3), (2,1), (3,3)\}$

$S = \{(1,2), (1,3), (2,1), (3,3)\}$

$S^2 = \{(1,1), (1,3), (2,2), (2,3), (3,3)\}$.

$R^2 = R \circ R$.

$R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}$

$R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}$.

$R^2 = \{(1,1), (1,3), (3,3), (3,1), (2,1), (1,2), (2,3), (3,2)\}$.

$S \circ R = \{(1,3), (1,1), (1,3), (2,1), (2,2), (3,1), (3,3)\}$.

$S - R = \{(1,3), (2,1)\}$.

3) consider the relation $R = \{(a,b), (b,c), (b,d), (d,a), (c,c)\}$ on $A = \{a,b,c,d\}$

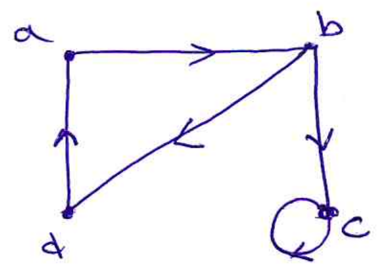
i) Draw a digraph for the relation R

ii) Draw a digraph for the complement of R.

iii) Draw a digraph for the inverse of R.

iv) Draw a digraph for $R \circ R^{-1}$.

i)

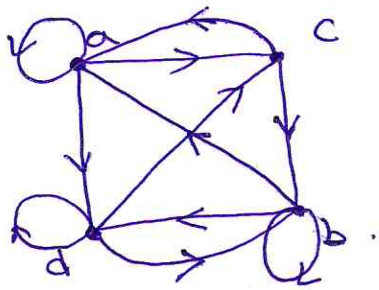


$\bar{R} = (A \times A) - R$.

$= \{(a,b) \mid (a,b) \notin R\}$.

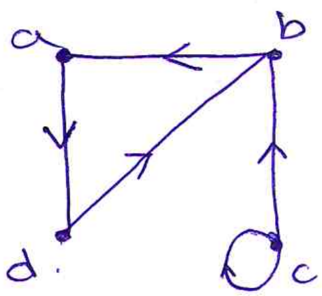
$A \times A = \{(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,a), (c,b), (c,c), (c,d), (d,a), (d,b), (d,c), (d,d)\}$.

$\bar{R} = \{(a,a), (a,c), (a,d), (b,a), (b,b), (c,a), (c,b), (c,d), (d,b), (d,c), (d,d)\}$.

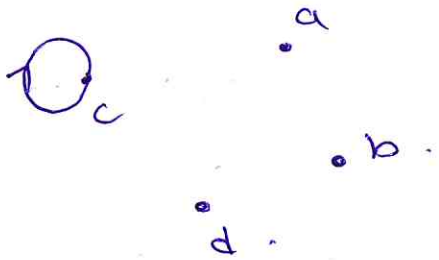


iii) $R = \{(a,b), (b,c), (b,d), (d,a), (c,c)\}$

$R^{-1} = \{(b,a), (c,b), (d,b), (a,d), (c,c)\}$.



iv) $R \cap R^{-1} = \{(c,c)\}$.



Suppose $R_1 \subseteq A \times B$ & $R_2 \subseteq B \times C$

The composition of R_1 & R_2 is denoted as $R_1 R_2$

(or) $R_1 \circ R_2$ & defined as

$$R_1 R_2 = \{(x,z) \,/\, (x,y) \in R_1 \ \& \ (y,z) \in R_2\}.$$

Eg:- $R_1 \subseteq A \times A$ where $A = \{1,2,3\}$

Let $R_1 = \{(1,1),(1,3),(2,1),(2,3),(3,2)\}$.

$R_1^2 = R_1 R_1$

$= \{(1,1),(1,3),(2,1),(2,3),(3,2)\}$.
$\{(1,1),(1,3),(2,1),(2,3),(3,2)\}$

$= \{(1,1),(1,3),(1,2),(2,1),(2,3),(2,2),(3,1),$
$(3,3)\}$

$R_1^3 = R_1^2 R_1$

$= \{(1,2),(1,1),(1,3),(2,1),(2,3),(3,1),$
$(3,3),(3,2)\}$.

Transitive closure:-

Let M_R be the relation matrix of a relation R on a set A of n elements. Then the transitive closure matrix M_{R^+} is given by

$$M_{R^+} = M_{R^1} \cup M_{R^2} \cup M_{R^3} \ldots \cup M_{R^n}.$$

The transitive closure of a relation R is the smallest transitive relation containing R. We denote transitive closure of R by R^+.

If A is any finite set containing 'n' elements & 'R' is a relation on A then

$$\boxed{R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots \cup R^n}$$

where $R^2 = R \cdot R$, $R^3 = R^2 \cdot R$, $\ldots$ $R^n = R^{n-1} \cdot R$

1) Let $A = \{1, 2, 3, 4\}$ &

$R = \{(1,2), (2,3), (3,4)\}$ be a relation on A. Then find R^+.

$$R^2 = R \cdot R = \{(1,2), (2,3), (3,4)\} \cdot \{(1,2), (2,3), (3,4)\}$$

$$= \{(1,3), (2,4)\}.$$

$R^3 = R^2 \cdot R$.

$= \{(1,3),(2,4)\} \cdot \{(1,2),(2,3),(3,4)\}$.

$= \{(1,4)\}$.

$R^4 = R^3 \cdot R$.

$= \{(1,4)\} \cdot \{(1,2),(2,3),(3,4)\}$

$= \emptyset$.

$\Rightarrow R^4 = R^5 = R^6 = \cdots = R^n = \emptyset$

$R^+ = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^n$.

$= \{\{(1,2),(2,3),(3,4)\} \cup \{(1,3),(2,4)\} \cup \{(1,4)\} \cup \emptyset \cdots \cup \emptyset\}$

$= \{(1,2),(2,3),(3,4),(1,3),(2,4),(1,4)\}$

The transitive reflexive closure of R is denoted as R^* & defined as.

$R^* = R^+ \cup \{(a,a) / a \in A\}$.

2) Let $A = \{a, b, c, d, e\}$ &

$R = \{(a,a), (a,b), (b,c), (c,d), (c,e), (d,e)\}$.

Find the transitive closure of R.

Solution:-

Given $R = \{(a,a), (a,b), (b,c), (c,d), (c,e), (d,e)\}$.

$R^2 = R \cdot R$.

$= \{(a,b), (a,c), (b,d), (c,e), (b,e), (a,a)\}$.

$R^3 = R^2 \cdot R$.

$= \{(a,b), (a,c), (b,d), (c,e), (b,e), (a,a)\} \cdot \{(a,a), (a,b), (b,c), (c,d), (c,e), (d,e)\}$.

$= \{(a,c), (a,d), (b,e), (a,e), (a,a), (a,b)\}$.

$R^4 = R^3 \cdot R$.

$= \{(a,a), (a,c), (a,d), (b,e), (a,e), (a,b)\}$.
$\{(a,a), (a,b), (b,c), (c,d), (c,e), (d,e)\}$

$= \{(a,a), (a,b), (a,d), (a,e), (a,c)\}$

$R^5 = R^4 \cdot R = \{(a,a), (a,b), (a,d), (a,e), (a,c)\}$.
$\{(a,a), (a,b), (b,c), (c,d), (c,e), (d,e)\}$

$= \{(a,a), (a,b), (a,c), (a,e), (a,d)\}$

$R^4 = R^5 = R^6 = \ldots$

$\therefore R^+ = R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots$

$= \{ (a,a), (a,b), (b,c), (c,d), (c,e), (d,e),$
$(a,c), (b,d), (b,e), (a,e) \}$.

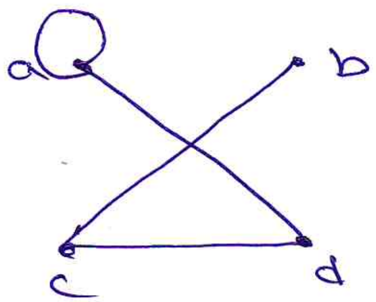
Adjacency matrix Representation.

Let $G=(V,E)$ be a graph with 'n' vertices ordered from v_1 to v_n. An $n \times n$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

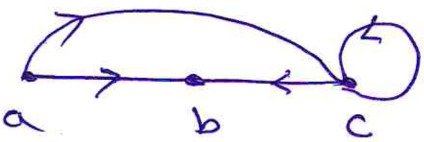
is called the adjacency matrix of G.
It is denoted by A_G.

1) Find the adjacency matrix of the following graph.



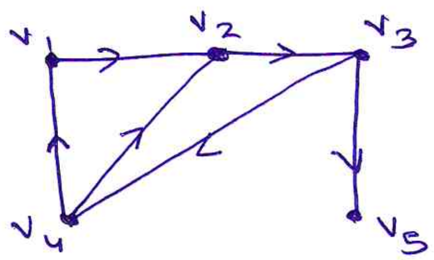
$$A_G = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

2) Write the adjacency matrix of the following digraph.



$$A_G = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 0 & 0 & 0 \\ c & 0 & 1 & 1 \end{array}$$

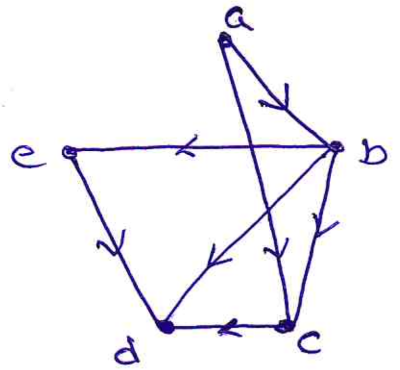
3) Find the adjacency matrix of the following digraph.

i) 

$R = \{(v_1, v_2), (v_2, v_3), (v_3, v_5), (v_4, v_1), (v_4, v_2), (v_3, v_4)\}$

$$
\begin{array}{c|ccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & 0 & 1 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 1 & 1 \\
v_4 & 1 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

ii)

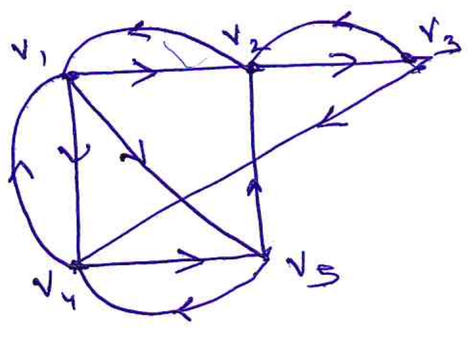


$R = \{(a,b), (b,c), (a,c), (b,e), (e,d), (b,d), (c,d)\}$.

$A_G =$

	a	b	c	d	e
a	0	1	1	0	0
b	0	0	1	1	1
c	0	0	0	1	0
d	0	0	0	0	0
e	0	0	0	1	0

iii)



4) Give the adjacency matrix of the digraph

a) $G = \{a, b, c, d, R\}$ where $R = \{(a,b), (b,c), (d,c), (d,a)\}$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b) Give the Boolean matrix representation of the transitive closure R^+.

$R = \{(a,b), (b,c), (d,c), (d,a)\}$

$R^2 = R \cdot R$.

$= \{(a,c), (d,b)\}$.

$R^3 = R^2 \cdot R$.

$= \{(d,c)\}$.

$R^4 = R^3 \cdot R$.

$= \emptyset$.

$R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n$.

$= \{(a,b), (b,c), (d,c), (d,a)\} \cup \{(a,c), (d,b)\} \cup$
$\{(d,c)\} \cup \emptyset \dots$

$= \{(a,b), (b,c), (d,c), (d,a), (a,c), (d,b)\}$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$$

c) Give the Boolean matrix representation of the transitive reflexive closure R^+.

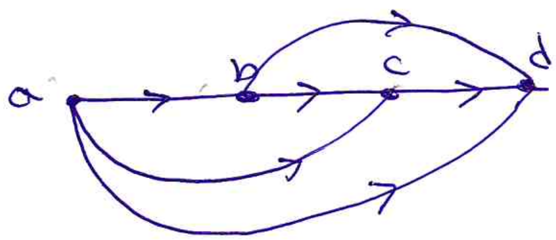
The transitive reflexive closure of R is denoted as R^* & defined as

$$R^* = R^+ \cup \{(a,a) / a \in A\}.$$

$$= \{(a,b), (b,c), (d,c), (d,a), (a,c), (d,b), (a,a), (b,b), (c,c), (d,d)\}$$

$$R^* = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 0 & 1 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$$

5) Give the adjacency matrix of the digraph shown in figure.



$R = \{(a,b), (b,c), (c,d), (b,d), (a,c), (a,d)\}$.

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) Give the Boolean matrix representation of the transitive closure of the relation represented by this digraph.

$R = \{(a,b), (b,c), (c,d), (b,d), (a,c), (a,d)\}$

$R^2 = R \cdot R$.

$= \{(a,c), (b,d), (a,d)\}$.

$R^3 = R^2 \cdot R$.

$= \{(a,d)\}$.

$R^4 = R^3 \cdot R$

$= \emptyset$.

$R^+ = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

$= \{(a,b), (b,c), (c,d), (b,d), (a,c), (a,d),$ ~~[illegible]~~$\}$

$$R^+ = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Warshall's Algorithm to find the transitive closure of a relation:

i) First transfer to M_K all 1's in M_{K-1}

ii) Record all positions $P_1, P_2, \ldots$ in column K of M_{K-1}, where the entry is 1, & the positions $a_{11}, a_{12}, \ldots$ in row K of M_{K-1}, where the entry is 1

iii) Put a 1 in each positions (P_i, a_{ij}) of M_K (Provided a 1 is not already there from a previous step).

1) Compute the adjacency matrix of R^+ using Warshall's algorithm.

1) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution :-

Let $M_0 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$

$\underline{C}$ $\underline{R}$

b b

(b, b).

$M_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 1 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$

C R

(a, b) (a, b, c)

{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)}.

$M_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 0 \\ b & 1 & 1 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$

C R

(a, b) d.

{(a, d), (b, d)}.

$$M_3 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$$

C	R
(a, b, c)	–

$R^+ = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (a,d), (b,d)\}$.

2)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let
$$M_0 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

C	R
1	(1, 2)

$\{(1,1), (1,2)\}$.

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C	R
1	3.

(1, 3).

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C   R

(1, 2)   (4, 5)

{(1, 4), (1, 5), (2, 4), (2, 5)}

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C   R

(1, 2, 3)   5.

{(1, 5), (2, 5), (3, 5)}

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C   R

(1, 2, 3, 4)   -

R^+ = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)}.

3) $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

let $M_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C (1, 2) R (1, 4).

{(1,1), (1,4), (2,1), (2,4)}.

$M_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & \cancel{0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C 2 R. (1, 2, 4)

{(2,1), (2,2), (2,4)}

$M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C 4 R –

$M_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

C (1, 2) R 3.

{(1,3), (2,3)}.

$$M_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R^+ = \{(1,1), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (4,3)\}$.

<u>By transitive closure method</u>

$R = \{(1,1), (1,4), (2,1), (2,2), (4,3)\}$

$R^2 = \{(1,4), (1,3), (2,4), (2,1)\}$.

$R^3 = R^2 \cdot R = \{(1,3), (2,3), (2,4), (2,1)\}$

$R^4 = R^3 \cdot R = \{(2,1), (2,3), (2,4)\}$

$R^5 = R^4 \cdot R = \{(2,1), (2,3), (2,4)\}$.

$\vdots$

$R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n$.

$= \{(1,1), (1,4), (2,1), (2,2), (4,3), (1,3), (2,3), (2,4)\}$.

4) Give the adjacency matrix of digraph $G = \{(a,b,c,d), R\}$ where $R = \{(a,b), (b,c), (d,c), (d,a)\}$. Give the Boolean matrix of transitive closure by using warshall's algorithm. ~~Give~~ Give the Boolean matrix representation of the transitive reflexive closure.

Solution :-

Given $R = \{(a,b), (b,c), (d,c), (d,a)\}$.

let $M_0 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 1 & 0 \end{array}$

C	R
d	b.
(d, b).	

Transfer all 1's from M_0 to M_1.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

C	R.
(a, d)	c
$\{(a,c), (d,c)\}$	

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

C	R
(a, b, d)	–

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

c R

– (a, b, c)

$$M_4 = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$R^+ = \{(a, b), (a, c), (b, c), (d, a), (d, b), (d, c)\}$.

Transitive reflexive closure R^* is denoted by R^*

$R^* = R^+ \cup \{(a, a) \mid a \in A\}$.

$= \{(a, b), (a, c), (b, c), (d, a), (d, b), (d, c),$
$(a, a), (b, b), (c, c), (d, d)\}$.

$$R^* = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

3) Let $A = \{a, b, c, d\}$ & R be a relation defined on A as $R = \{(a,a), (a,b), (b,c), (c,d), (c,c), (d,e)\}$. Find transitive closure using warshall's Algorithms

solution :- Given $R = \{(a,a), (a,b), (b,c), (c,d), (c,c), (d,e)\}$.

$$\text{let } M_0 = \begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \phi & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

C   R

a   (a, b)

$\{(a,a)\ (a,b)\}$.

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C   R

a   c

(a, c).

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C   R

(a, b, c)   (c, d).

$\{(a,c), (a,d), (b,c), (b,d), (c,c), (c,d)\}$.

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

C R.

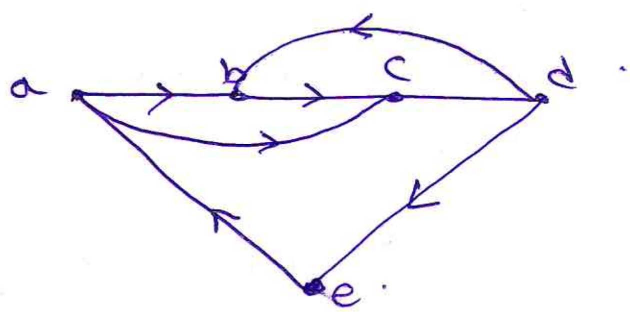
(a, b, c) e.

{(a, e), (b, e), (c, e)}.

$$M_4 = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

R^+ = {(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, e)}.

6) Using warshall's algorithm, compute the adjacency matrix of the transitive closure of the digraph in figure.



Solution :-

$$
\text{Let } M_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

C   R

e   (b, c)

{(e, b), (e, c)}

$$
M_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}
$$

C   R

(a, d, e)   c

{(a, c), (d, e), (e, c)}

$$
M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}
$$

C   R

(a, b, d, e)   d

{(a, d), (b, d), (d, d) (e, d)}

$$
M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}
$$

C   R.

(a, b, c, d, e)   (b, c, d, e)

{(a, b), (a, c), (a, d), (a, e),
(b, b), (b, c), (b, d), (b, e),
(c, b), (c, c), (c, d), (c, e),
(d, b), (d, c), (d, d), (d, e),
(e, b), (e, c), (e, d), (e, e)}

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

C: (a, b, c, d, e) R: (b, c, d, e)

$$M_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Problems

Using warshall's algorithm compute the adjacency matrix of the transitive closure of the digraph $G = \{(a, b, c, d, e), R\}$ where $R = \{(a,b), (b,c), (c,d), (d,e), (e,d)\}$.

Compute the adjacency matrix of R^+ using warshall's Algorithm

a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

7) let $A = \{1, 2, 3, 4\}$ & let R be a relation on A defined by $R = \{(1,1), (1,2), (2,4), (3,2), (4,3)\}$. Find the transitive closure of R.

solution :-

$R^2 = R \cdot R = \{(1,2), (1,4), (2,3)\}$

$R^3 = R^2 \cdot R = \{(1,4), (1,3), (2,2)\}$

$R^4 = R^3 \cdot R = \{(1,3), (1,2), (2,4)\}$

$R^5 = R^4 \cdot R = \{(1,2), (1,4), (2,3)\}$.

$R^6 = R^5 \cdot R = \{(1,4), (1,3), (2,2)\}$

$R^7 = R^6 \cdot R = \{(1,3), (1,2), (1,4)\}$.

$R^8 = R^7 \cdot R = \{(1,2), (1,4), (1,3)\}$.

$R^7 = R^8 = R^9 = \ldots$

$R^+ = R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots R^n$

$= \{(1,1), (1,2), (2,4), (3,2), (4,3)\} \cup \{(1,2), (1,4), (2,3)\} \cup \{(1,4), (1,3), (2,2)\} \cup \{(1,3), (1,2), (2,4)\} \cup \{(1,2), (1,4), (2,3)\}$

$\ldots$

$R^+ = \{(1,1), (1,2), (2,4), (3,2), (4,3), (1,4), (2,3), (1,3), (2,2)\}$

A graph G is a pair of sets (V, E) where V is a set of vertices & E is a set of edges. If G is a directed graph (digraph) the elements of E are ordered pairs of vertices.

A graph with no loops is said to be simple graph or loop-free.

Directed path :- A directed path in a graph G (V, E) is a sequence of edges $e_1, e_2 \ldots e_n$ in E such that e_i (v_{i-1}, v_i) $i = 1, 2, \ldots n$. Here v_0 & v_n are called end points of the path.

The length of a path is defined as the number of edges involved in the path.

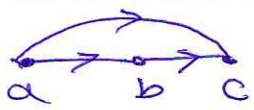
If the elements of E are ordered pairs denoted as (u, v) then G is called a directed graph (digraph).

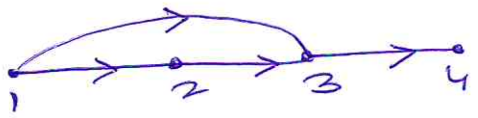
If the elements of E are unordered pairs denoted as {u, v} then G is called a non-directed graph.

Eg:- $G = (V, E)$

where $V = \{a, b, c\}$ & $E = \{(a, b), (b, c), (a, c)\}$.

is a directed graph.





Directed Path $= \{(1, 2), (2, 3), (3, 4)\}$.

Non-directed path $= \{(1, 2), (2, 3), (3, 1)\}$.

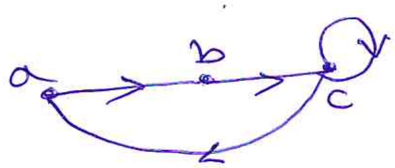
vertices 1, 4 are end Points of the Path.

A Path is simple if all edges & vertices on the path are distinct except the end points may be equal.

A path of length ≥ 1 with no repeated edges & whose end points are equal is a circuit.

A simple circuit is called a cycle.

Eg:-



$(a, b), (b, c), (c, a)$ - cycle.

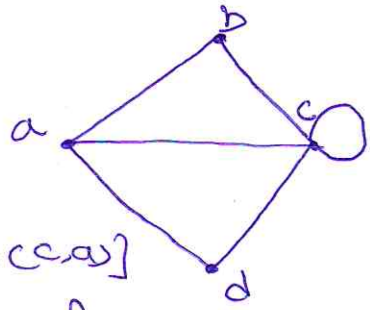
$(a, b), (b, c), (c, c), (c, a)$ - circuit.

$(a, b), (b, c)$ - Simple path.

Note:-

i) Every circuit is need not be a simple path

ii) every simple path need not be a cycle.



The path {c, c} is a cycle of length 1.

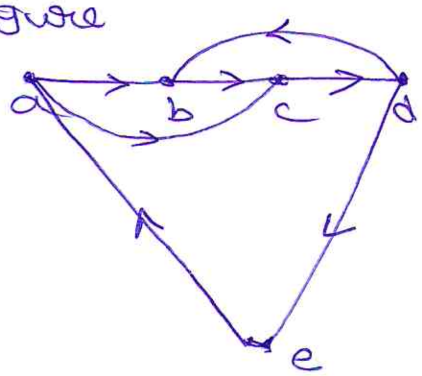
The sequence of edges {(a, b), (b, c), (c, a)} & {(a, d), (d, c), (c, a)} form cycles of length 3.

The path {(a, b), (b, c), (c, d), (d, a)} is a cycle of length 4.

{(a, b), (b, c), (c, c), (c, a)} is a circuit of the length 4 it is not a cycle because the sequence of vertices a - b - c - c - a includes more than one repeated vertex.

The sequence of edges {(a, b), (b, c), (c, a), (a, d), (d, c), (c, a)} forms a closed path of length 6 but this path is not a circuit because the edge {(c, a)} is repeated twice.

Consider the digraph in figure



a) Find all the simple directed paths from a to d.

{(a, b), (b, c), (c, d)}.

{(a, c), (c, d)}.

b) Find all of the simple non-directed paths from a to d.

{(a,b) , (d,b)}

{(a,c) , (b,c) , (d,b)}

{(e,a) , (d,e)}.

c) Find all of the directed cycles that start at d.

(d,b) , (b,c) , (c,d).

(d,e) , (e,a) , (a,b) , (b,c) , (c,d).

(d,e) , (e,a) , (a,c) , (c,d).

d) Find all of the non-directed cycles that start at d.

{(d,e) , (e,a) , (a,b) , (d,b)}

{(d,e) , (e,a) , (a,c) , (b,c) , (d,b)}.

{(d,b) , (a,b) , (a,c) , (c,d)}.

{(d,b) , (a,b) , (e,a) , (d,e)}.

{(d,b) , (b,c) , (a,c) , (c,d)}

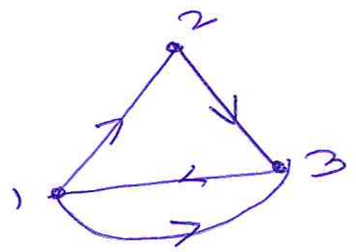
(d,b) , (b,c) , (a,c) , (e,a) , (d,e).

(d,b) , (a,b) , (a,c) , (c,d)}.

A pair of vertices in a digraph is said to be weakly connected if there is a non-directed path between them.

A pair of vertices in a digraph is said to be unilaterally connect if there is a directed path between them.

A pair of vertices x & y is said to be strongly connected. if there is a directed path from x to y & also from y to x.



(1, 3) - strongly connected.

(1, 2) - unilaterally connected.

(2, 3) - "

(3, 2) - weakly connected.