DISCRETE MATHEMATICS

Mathematical Logic

Introduction

Many proofs in Mathematics and many algorithms in computer science use logical expressions such as "If P then Q". We use different types of sentences say declarative,interrogative, exclamatory and imperative to express our ideas. But in mathematics, sentences which are declarative and about which it is possible to judge, whether they are true or false but not both are used to draw conclusions and to prove theorems. Such sentences are statements. This value associated with the truthfulness or falsity of a statement is called its truth values.

A Proposition (Statement): It is a well defined argument, which is either true or false, but not both.

Ex:- 1) Chennai is in Tamil Nadu.

2) 5+4=9

3) Close the door.

4) Where are you going (or) what are you doing?

1,2 are Statements, where as 3,4 are not statements: neither true nor false.

Statements are denoted by P,Q,R,… (or) p,q,r, …

Note:- T, F will be used for True and False respectively. Sometimes use 1, 0 for T, F.

Atomic Statement:- Any statement which do not contain any of the connectives is called atomic statement. It is also called as primary (or) primitive statement.

Connectives:-

There are five basic connectives which are frequently used

S.No.	Symbol	Name	Connective word
1	$\sim$ or $\neg$	Negation	Not
2	$\wedge$	Conjunction	And
3	$\vee$	Disjunction	Or
4	$\rightarrow$	Implication (or) Conditional	Implies or If….then

5	$\leftrightarrow$	**Bi-conditional**	**If and only if**

Negation: The negation of a statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the statement with the phrase " It is not the case that or it is not true that".

If P denotes a statement, then the negation of P is written as $\sim$P & read as "not P".

If the truth value of P is T, then the truth value of $\sim$P is F. Also, if the truth value of P is F, then the truth value of $\sim$P is T.

Truth Table for negation:

P	**$\sim$P**
T	**F**
F	**T**

Ex:- P: Nikhil is playing badminton
$\sim$P: Nikhil is not playing badminton.
Or $\sim$P: It is not the case that Nikhil is playing badminton.

Ex: P: There are 12 months in a year.
$\sim$P: There are not 12 months in a year.
Or $\sim$P: It is not true that there are 12 months in a year.

Note:- Alternative symbols that can be used to represent negation are "$\sim$" , "a bar" ($\bar{\ }$) or "Not".

Thus, $\urcorner$ P is written as $\sim$P or $\bar{P}$ or Not P

Conjunction:- Conjunction of two statements P and Q is denoted by the statement "$P \wedge Q$", which read as " P and Q" . The statement $P \wedge Q$ have truth value T whenever both P and Q have the truth value T otherwise it has truth value F.

Truth table for conjunction:

P	**Q**	$P \wedge Q$
T	T	T

T	F	F
F	T	F
F	F	F

Note: A is symmetric.

$\therefore P \wedge Q, Q \wedge P$ have same truth values.

Ex: P: Today is holiday.

Q: There are fruits in this room

$P \wedge Q$: Today is holiday & there are fruits in this room.

Ex: P: It is raining today , Q: There are 20 tables in this room.

$P \wedge Q$: It is raining today and there are 20 tables in this room

Disjunction:- Disjunction of two statements P and Q may be denoted by the statement $P \vee Q$, which is read as "P or Q" (or) P Join Q. The truth value of $P \vee Q$ is true whenever P is true (or) Q is true (or) both P, Q are true, Otherwise $P \vee Q$ is False.

Truth Table for Disjunction:

P	**Q**	**$P \vee Q$**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Note: Disjunction is Symmetric. i.e., $P \vee Q, Q \vee P$ have same truth values.

Ex: I shall either watch the cricket on TV or go to the stadium.

Molecular (compound or composite) Statements:

Those statements which contain one or more atomic statements and some connectives are called Molecular Statements.

Ex: Let P, Q be any two statements, Some of the molecular statements formed by using P & Q are $\sim \mathrm{P}, P \wedge Q, (P \wedge Q) \vee (\sim \mathrm{P})$

Conditional Statement:

If P & Q are any two statements, the statement $P \rightarrow Q$ which we read "If P then Q" is called a conditional statement or implication statement. The statement $P \rightarrow Q$ has a truth value F when Q has the truth value F & P has the truth value T, otherwise it has the Truth Value T.

Truth table for conditional statement:

P	Q	$\mathbf{P \rightarrow Q}$
T	T	T
T	F	F
F	T	T
F	F	T

Here the statement P is called **antecedent** & Q is called **consequent** in $P \rightarrow Q$.

Ex: If I get the book then I begin to read.
P: I get the book, Q: I began to read
Symbolic form is $\mathbf{P \rightarrow Q}$

Note: The Statement $P \rightarrow Q$ may be read as
(i) If P, then Q ii) Q if P iii) Q is necessary for P
iv) P is sufficient for Q v) P only if Q vi) P implies Q
vii) Q whenever P

Write the following statement in symbolic form.

Statement: If either John prefers tea or Jim prefers Coffee, then Rita prefers milk.
Solution:
Let A: John prefers tea, B: Jim prefers Coffee,C: Rita prefers milk.
Symbolic form is (AVB) →C

Statement: The crop will be destroyed if there is a flood.

Solution:

Let the statements be denoted as

A: The crop will be destroyed

B: There is a flood

It is better to rewrite the given statement as "If there is a flood, then the crop will be destroyed". Now it is easy to symbolize it as $B \rightarrow A$

Statement: If the sun is shining today, then 2+3 > 4.

Solution: Let P: The sun is shinig today, Q:2+3 > 4

In symbolic form is $P \rightarrow Q$

Bi-conditional Statements:

If P & Q are any two statements, then the statement $P \leftrightarrow Q$, which is read as "P if and only if Q", is called a biconditional statement. The statement $P \leftrightarrow Q$ has truth value T whenever both P & Q have identical truth values; otherwise its truth value is F.

Truth table for Bi-conditional statements:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: The statement $P \leftrightarrow Q$ may be read as

i) P if and only if Q

ii) If P then Q & conversely

iii) P is necessary & sufficient for Q.

Ex: Vijayawada is in A.P. if and only if 9 ÷ 3 =3

Let P:Ravi is rich, Q:Ravi is happy

Write each of the following in symbolic form

i) Ravi is poor but happy - $\urcorner P \wedge Q$

ii) Ravi is neither rich nor happy- $\urcorner P \wedge \urcorner Q$

iii) Ravi is rich and unhappy- $P \wedge \urcorner Q$

Write the symbolic statement of "If Rita and Sita go to I.T camp & Jim and John go to P.C camp then the college gets the good name".

Let A: Rita goes to I.T camp
B: Sita goes to I.T Camp
C: Jim goes to PC Camp
D: John goes to PC Camp
E: College gets good name

Symbolic form of given statement is:
$((A \wedge B) \wedge (C \wedge D)) \longrightarrow E$

Let P: Ramu reads Newsweek, Q: Ramu reads the New Yorker, R:Ramu reads Time

Write each of the following in symbolic form:

1) Ramu reads Newsweek or the New Yorker but not Time.
2) Ramu reads Newsweek and the New Yorker, or he does not read Newsweek and Time.
3) It is not true that Ramu reads the New Yorker but not Time.
4) It is not true that Ramu reads Time or the New Yorker but not Newsweek.

Solution:

1) $(P \vee Q) \wedge \sim R$

2) $(P \wedge Q) \vee (\sim(Q \wedge R))$

3) $\sim(P \wedge \sim R)$

4) $\sim((R \vee Q) \wedge \sim P)$

Demorgans Laws:

$\sim (P \vee Q) = \sim p \wedge \sim Q$

$\sim (P \wedge Q) = \sim p \ \vee \sim Q$

Use Demorgan's laws to write the negation of each statement

i) I want a car and a worth cycle.
I don't want a car or not a worth cycle.

ii) My cat stays outside or it makes a mess.

My cat not stays outside and it doesn't make a mess.

iii) I' ve fallen and I can't get up.
I don't have fallen or I can get up.

iv) You stay or you don't get a good grade.
You don't stay or you can get a good grade.

Write the negation of the following statements:

1) Keerthi will take a job in industry or go to graduate school.
2) James will bicycle or run tomorrow.
3) If the processor is fast, then the printer is slow.

Solution:

1) Keerthi will not take a job in industry and not go to graduate school.
2) James will not bicycle and not run tomorrow.
3) The processor is fast and the printer is not slow.

Well-formed formulas:

A string of symbols containing the statement variables, connectives and parenthesis is said to be well-formed formula if it can be obtained by finitely many applications of the rules $R_1, R_2 \& R_2$

R_1: A statement variable standing alone is a well-formed formula.

R_2: If A is a well-formed formula, then ~A is a well-formed formula.

R_3: If A & B are well-formed formulas, then (A∧ B), (A∨ B), (A→ B) & (A⟷ B) are well formed formulas.

Ex: -1) P∨Q is not well-formed formula but (P→Q) is a well-formed formula.

2)($P \to Q$)→(∧Q) is not well-formed formula because ∧Q is not statement variable.

3) ($P \to Q$ is not well-formed formed but ($P \to Q$) is a well-formed formula.

Construction of Truth Tables:

A Truth Table consists of columns and rows. The number of columns depend upon the numbers of sample propositions and connectives used to form a compound proposition. The number of rows in a truth table are found on the basis of simple propositions.

For example, if there are two simple propositions, there will be 2^2=4 rows. For n simple propositions, the total number of rows will be $\mathbf{2^n}$.

Construct the Truth table for $P \wedge (P \longrightarrow Q)$

P	Q	$P \longrightarrow Q$	$P \wedge (P \longrightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Construct the Truth table for $\lceil P \wedge Q$, $\lceil P \vee Q$

P	Q	$\lceil P$	$\lceil P \wedge Q$	$\lceil P \vee Q$
T	T	F	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	F	T

Construct the Truth table for $[(P \wedge Q) \vee \lceil R] \longleftrightarrow P$

P	Q	R	$P \wedge Q$	$\lceil R$	$[(P \wedge Q) \vee \lceil R$	$[(P \wedge Q) \vee \lceil R] \leftrightarrow P$
T	T	T	T	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	F	F	F	T
F	T	F	F	T	T	F
F	F	T	F	F	F	T
F	F	F	F	T	T	F

Construct the Truth Table for $(P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P)$

P	Q	R	$(P \wedge Q)$	$(Q \wedge R)$	$(R \wedge P)$	$(P \wedge Q) \vee (Q \wedge R)$	$(P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	F	F	F	F
F	T	T	F	T	F	T	T
F	T	F	F	F	F	F	F

F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Construct the Truth Table for $((P \vee Q) \wedge \lceil R)) \leftrightarrow (Q \longrightarrow R)$

P	Q	R	$P \vee Q$	$\lceil R$	$(P \vee Q) \wedge \lceil R)$	$Q \longrightarrow R$	$((P \vee Q) \wedge \lceil R)) \leftrightarrow (Q \longrightarrow R)$
T	T	T	T	F	F	T	F
T	T	F	T	T	T	F	F
T	F	T	T	F	F	T	F
T	F	F	T	T	T	T	T
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	T	F	T	F

Construct the Truth Table for **$(P \longrightarrow Q) \wedge (Q \longrightarrow R)$**

P	Q	R	$P \longrightarrow Q$	$Q \longrightarrow R$	$(P \longrightarrow Q) \wedge (Q \longrightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Construct the Truth Table for **$\sim P \wedge (P \longrightarrow Q)$**

P	Q	$\sim P$	$P \longrightarrow Q$	$\sim P \wedge (P \longrightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

Construct the Truth Table for $(P \wedge (Q \longrightarrow R)) \longrightarrow (Q \longrightarrow R)$

Solution:

P	Q	R	Q⟶ R	P ∧ (Q⟶ R))	(P ∧ ((Q⟶ R)) ⟶ (Q⟶ R)
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Two logical expressions are said to be equivalent if they have the same truth value in all cases. Sometimes this fact helps in proving a mathematical result by replacing one expression with another equivalent expression, without changing the truth value of the original compound proposition.

Types of propositions based on Truth values

There are three types of propositions when classified according to their truth values

1. Tautology – A proposition which is always true, is called a tautology.
2. Contradiction – A proposition which is always false, is called a contradiction.
3. Contingency – A proposition that is neither a tautology nor a contradiction is called a contingency

Tautology:-

A statement formula that is true regardless of the truth values of the statements that replace the variables in it is called a Universally valid formula (or) a Tautology (or) a logical truth.
The formula **PV~P** represents a Tautology.

P	~P	PV~P
T	F	T
F	T	T

Note: -If all the entries in the last column of a table are T, then given formula is a Tautology.

Contradiction:- A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called a contradiction.

The Formula $P\wedge\sim P$ represents a contradiction.

P	~P	$P\wedge\sim P$
T	F	F
F	T	F

Note:- i) $P\vee\sim P$ is a Tautology. Since it always has truth value T

i) $P\wedge\sim P$ is a contradiction since it always has truth value F.

Contingency:- A proposition that is neither a Tautology nor a contradiction is called a Contingency.

Write the statement in the symbolic form & find whether it is a Tautology or not.
If I am hungry & thirsty, then I am hungry.
Solution:
Let P: I am hungry Q: I am thirsty
In symbolic form $(P\wedge Q)\rightarrow P$

P	Q	$P\wedge Q$	$(P\wedge Q)\rightarrow P$
T	T	T	T
T	F	F	T

F	T	F	T
F	F	F	T

So it is a Tautology.

Check whether the following formulas are tautologies or not $(P \vee Q) \rightarrow P$

P	**Q**	**PVQ**	**$(P \vee Q) \rightarrow P$**
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since all the entries in the last column of the truth table of $(P \vee Q) \rightarrow P$ are not T, the formula is not a tautology.

Check whether $P \vee [\neg(P \wedge Q)]$ is a Tautology

P	**Q**	**$P \wedge Q$**	**$\neg(P \wedge Q)$**	**$P \vee [\neg(P \wedge Q)]$**
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since all the entries in the last column of the truth table are T,the formula is a Tautology.

Check whether the formula $P \longrightarrow (P \vee Q)$ is a Tautology or not.

P	**Q**	**PVQ**	**$P \longrightarrow (P \vee Q)$**
T	T	T	T
T	F	T	T
F	T	T	T

F	F	F	T

Prove that $\sim(P \vee \sim Q) \rightarrow \sim P$ is a Tautology.

P	Q	~P	~Q	$P \vee \sim Q$	$\sim(P \vee \sim Q)$	$\sim(P \vee \sim Q) \rightarrow \sim P$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	T	F	T

Therefore the given formula is a Tautology.

Prove that $P \wedge (Q \vee R) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$ is a Tautology.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$	$P \wedge (Q \vee R) \leftrightarrow ((P \wedge Q) \vee (P \wedge R))$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	T
T	F	T	T	T	F	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

Show that the proposition $(P \vee \sim Q) \wedge (\sim P \vee \sim Q) \vee Q$ is a Tautology.

P	Q	~P	~Q	$P \vee \sim Q$	$\sim P \vee \sim Q$	$(P \vee \sim Q) \wedge (\sim P \vee \sim Q)$	$(P \vee \sim Q) \wedge (\sim P \vee \sim Q) \vee Q$
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	T	T	T

Since all the entries in the last column of the truth table are T,the formula is a Tautology.

EQUIVALENCE OF FORMULAS

Two formulas A & B are said to be equivalent to each other if and only if $A \leftrightarrow$ B is a Tautology.

The equivalence of two formulas A & B is denoted by $A \Leftrightarrow$B, which is read as A is equivalent to B.

<u>Note:-</u> $\Leftrightarrow$ is only symbol, but not connective.

A $\Leftrightarrow$B if and only if the truth values of A, B are same.
i.e., the final columns in the truth tables of A, B are same
i.e., truth tables of A,B are same.
Equivalence relation is symmetric & transitive.

Ex: i) $\sim(\sim P)$ is equivalent to P
ii) $P \wedge P$ is equivalent to P
iii) $P \vee \sim P$ is equivalent to $Q \vee \sim Q$
iv) $(P \wedge \sim P) \vee Q$ is equivalent to Q

Method I(truth Table Method):
One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

Prove the following:
1) $(P \wedge \sim P) \vee Q \Leftrightarrow Q$

P	Q	~P	P∧~P	(P∧~P)VQ
T	T	F	F	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

Since truth values are same, therefore $(P \wedge \sim P) \vee Q \Leftrightarrow Q$

2) $(P \rightarrow Q) \Leftrightarrow (\sim P \vee Q)$

P	Q	~P	~PVQ	$P \rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since truth values are same, therefore $(P \rightarrow Q) \Leftrightarrow (\sim P \vee Q)$

Show that **$P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are Logically Equivalent.**

Solution:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the truth values of given formulas are same, therefore they are Logically Equivalent.

Show that $P \rightarrow (Q \rightarrow P)$ & $\sim P \rightarrow (P \rightarrow Q)$ are logically equivalent.

P	Q	$Q \rightarrow P$	$P \rightarrow (Q \rightarrow P)$	$P \rightarrow Q$	~P	$\sim P \rightarrow (P \rightarrow Q)$
T	T	T	T	T	F	T
T	F	T	T	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

Since the truth values of given formulas are same, therefore they are Logically Equivalent.

Method II (Replacement Process)

In this method ,we replace any part of a statement formula by another equivalent formula

Ex: $P \to (Q \to R) \Leftrightarrow P \to (\sim Q \vee R)$

Equivalent Formulas:

Let P,Q,R be any 3 statements, Then all possible formulas may be written as:

1.	$P \vee P \Leftrightarrow P$; $P \wedge P \Leftrightarrow P$	Idempotent Laws
2.	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative Laws
3.	$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$	Commutative Laws
4.	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive Laws
5.	$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow P$	Identity Laws
6.	$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow F$	Domination Laws
7.	$\sim (\sim P) \Leftrightarrow P$	Double Negation Laws
8.	$\sim (P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$ $\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$	De Morgan's Laws
9.	$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$	Absorption Laws
10.	$P \vee \sim P \Leftrightarrow T$ $P \wedge \sim P \Leftrightarrow F$	Negation Laws
11.	$P \to Q \Leftrightarrow \sim P \vee Q$ $Q \to R \Leftrightarrow \sim Q \vee R$	Laws of Implication

Sl.No	Logical Equivalence involving implications
1	$P \rightarrow Q \Leftrightarrow \sim P \vee Q$
2	$P \rightarrow Q \Leftrightarrow \sim Q \rightarrow \sim P$
3	$P \vee Q \Leftrightarrow \sim P \rightarrow Q$
4	$P \wedge Q \Leftrightarrow \sim(P \rightarrow \sim Q)$
5	$\sim(P \rightarrow Q) \Leftrightarrow p \wedge \sim Q$
6	$(P \rightarrow Q) \wedge (P \rightarrow R) \Leftrightarrow P \rightarrow (Q \wedge R)$
7	$(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$
8	$(P \rightarrow Q) \vee (P \rightarrow R) \Leftrightarrow P \rightarrow (Q \vee R)$
9	$(P \rightarrow R) \vee (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$
10	$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
11	$P \leftrightarrow Q \Leftrightarrow \sim P \leftrightarrow \sim Q$
12	$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$
13	$\sim(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \sim Q$

1) Show that $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\sim Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Solution:

$P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\sim Q \vee R)$ (Law of implication)

$\Leftrightarrow \sim P \vee (\sim Q \vee R)$ (Law of implication)

$\Leftrightarrow (\sim P \vee \sim Q) \vee R$ (Associative Law)

$\Leftrightarrow \sim(P \wedge Q) \vee R$ (Demorgans laws)

$\Leftrightarrow (P \wedge Q) \rightarrow R$ (Law of implication)

Hence proved.

Show the following equivalences:

2) a) $P \rightarrow (Q \rightarrow P) \Leftrightarrow \sim P \rightarrow (P \rightarrow Q)$

Solution:

$P \rightarrow (Q \rightarrow P)$ $\Leftrightarrow P \rightarrow (\sim Q \vee P)$ (by law of	$\sim P \rightarrow (P \rightarrow Q)$ $\Leftrightarrow \sim(\sim P) \vee (P \rightarrow Q)$ (by law of

implication) ⇔ ~P V (~Q V P) (by law of implication) ⇔ (~P V ~Q) V P (associative law) ⇔ P V (~P V ~Q)(commutative law) ⇔ (P V ~P) V ~Q (associative law) ⇔ T V ~Q ⇔ T (domination law)	implication) ⇔ P V (~P V Q) (by law of implication) ⇔ (P V ~P) V Q (associative law) ⇔ T V Q (P V ~P =T Negation law) ⇔ T (domination law)

Therefore, the two given formulas are equivalent.

b) P → (Q V R) ⇔ (P → Q) V (P → R)

Solution:-

P → (QV R) ⇔ ~P V (QV R) (by law of implication) ⇔ (~P V Q) V R (associative law) (P → Q) V (P → R) ⇔ (~P V Q) V (~P V R) (by law of implication) ⇔ (~P V Q) V R (associative law) Therefore, the two given formulas are equivalent.	(or)	P → (Q V R) ⇔ ~P V (QV R) (by law of implication) ⇔ (~P V ~P) V (QV R) (*PVP*⇔P Idempotent Law) ⇔ (~P V Q) V (~P V R) (associative law) ⇔ (P → Q) V (P → R) (by law of implication)

3) Show that [~P∧(~Q∧R)]V[(Q∧R)V(P∧R)] ⇔ R

Solution:-

Let us consider [~P∧(~Q∧R)]V[(Q∧R)V(P∧R)]

⇔[(~P∧~Q)∧R)]V[(Q∧R)V(P∧R)] (Associative Law)

⇔[~(PVQ)∧R]V[(QVP)∧R] (Demorgans law & Distributive law)

⇔[~(PVQ)∧R]V[(PVQ)∧R] (Commutative law)

⇔[~(PVQ)V(PVQ)]∧R (Distributive law)

⇔ T ∧ R (~PVP=T Negation law)

⇔ R (Identity law)

Hence proved.

4)Show that ((PVQ) ∧ ~(~P ∧ (~QV~R))) V (~P∧~Q) V (~P∧~R) is a tautology.

Solution:-

By using Demorgans laws, We have

~P ∧ ~Q ⇔ ~(P V Q) and ~P ∧ ~R ⇔ ~(P V R)

Therefore (~P∧~Q) V (~P∧~R) ⇔ ~(P V Q) V~(P V R) ⇔ ~((P V Q) ∧ (P V R))

& $\sim(\sim P \land (\sim Q \lor \sim R))$

$\Leftrightarrow \sim(\sim P \land \sim(Q \land R)$ (Demorgans law)

$\Leftrightarrow P \lor (Q \land R)$ (Demorgans law)

$\Leftrightarrow (P \lor Q) \land (P \lor R)$(Distributive law)

Now, the given formula is equivalent to

$((P \lor Q) \land (P \lor Q) \land (P \lor R))\lor \sim((P \lor Q) \land (P \lor R))$

$\Leftrightarrow((P \lor Q) \land (P \lor R)) \lor \sim((P \lor Q) \land (P \lor R))$

$\Leftrightarrow$ T (since $P \lor \sim P \Leftrightarrow T$)

Therefore given formula is a tautology.

5)Show that $\sim(P\land Q) \rightarrow \sim P \lor (\sim P \lor Q)) \Leftrightarrow (\sim P \lor Q)$

Solution:-

$\sim(P\land Q) \rightarrow (\sim P \lor (\sim P \lor Q))$

$\Leftrightarrow (P\land Q) \lor (\sim P \lor (\sim P \lor Q))$ (by law of implication)

$\Leftrightarrow (P\land Q) \lor (\sim P \lor Q)$ (since $\sim P \lor \sim P \Leftrightarrow \sim P$)

$\Leftrightarrow (\sim P \lor Q) \lor (P\land Q)$ (Commutative law)

$\Leftrightarrow (\sim P \lor Q \lor P) \land (\sim P \lor Q \lor Q)$ (Distributive law)

$\Leftrightarrow (T \lor Q) \land (\sim P \lor Q)$ ($P\lor\sim P\Leftrightarrow T$ Negation law, $Q \lor Q=Q$ Idempotent Law)

$\Leftrightarrow T \land (\sim P \lor Q)$ (Domination Law $T \lor Q=Q$)

$\Leftrightarrow (\sim P \lor Q)$ (Identity law $T \land P=P$)

Hence Proved

6)Show that $(P \lor Q) \land (\sim P \land (\sim P \land Q)) \Leftrightarrow (\sim P \land Q)$

Solution:-

$(P \lor Q) \land (\sim P \land (\sim P \land Q))$

$\Leftrightarrow (P \lor Q) \land (\sim P \land Q)$ (since $\sim P \land \sim P = \sim P$ Idempotent Laws)

$\Leftrightarrow (P \land \sim P) \lor (Q \land \sim P) \land Q$ (since $P \land \sim P = F$ Negation Laws)

$\Leftrightarrow F \lor (Q \land \sim P)$

$\Leftrightarrow (Q \land \sim P)$ (Identity Laws $P \lor F \Leftrightarrow P$)

$\Leftrightarrow (\sim P \land Q)$ (commutative law $P\land Q \Leftrightarrow Q\land P$)

Hence proved.

7) Are $(P \rightarrow Q)\rightarrow R)$ & $P \rightarrow (Q \rightarrow R)$ are logically equivalent. Justify your answer by using rules of logic to simplify both expressions and also by using the truth tables.

Solution:-

Let us consider $(P \rightarrow Q)\rightarrow R$

$\Leftrightarrow (\sim P \lor Q) \rightarrow R$ (by law of implication)

$\Leftrightarrow \sim(\sim P \lor Q) \lor R$ (by law of implication)

$\Leftrightarrow (P \vee \sim Q) \vee R$ (Associative Law)……………….(1)
Let us consider $P \rightarrow (Q \rightarrow R)$
$\Leftrightarrow P \rightarrow (\sim Q \vee R)$ (by law of implication)
$\Leftrightarrow \sim P \vee (\sim Q \vee R)$ (by law of implication)
$\Leftrightarrow (\sim P \vee \sim Q) \vee R$ (Associative Law)…………..(2)

1) From (1) & (2) the given two formulas are not logically equivalent.

By using the truth tables

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

The two formulas are not equivalent

Duality Law:

Two formulas A and A* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other. If the formulae, A contains the special variable T or F, then A*, its duals, is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Write the duals of

1)$(P \vee Q) \wedge R$

Solution: Dual of given formula is $(P \wedge Q) \vee R$

2)$(P \wedge Q) \wedge T$

Solution: Dual of given formula is $(P \vee Q) \vee F$

3)If the formula A is given by

A: $\sim(P \vee Q) \wedge (P \vee \sim(Q \wedge \sim R))$, then find A*

Solution: Dual of given formula is A*: $\sim(P \wedge Q) \vee (P \wedge \sim(Q \vee \sim R))$

TAUTOLOGICAL IMPLICATIONS

The proposition $\boldsymbol{Q} \rightarrow \boldsymbol{P}$ is called the converse of $\boldsymbol{P} \rightarrow \boldsymbol{Q}$.
The proposition of $\sim \mathbf{Q} \rightarrow \sim \mathbf{P}$ is called the contra positive of $\boldsymbol{P} \rightarrow \boldsymbol{Q}$
The proposition of $\sim \mathbf{P} \rightarrow \sim \mathbf{Q}$ is called the inverse of $\boldsymbol{P} \rightarrow \boldsymbol{Q}$

Functionally complete Set: Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from this set is called a functionally complete set of connectives.

NAND:- The word NAND is a combination of NOT & AND, where NOT Stands for Negation & AND for conjunction. The connective NAND is denoted by the symbol ↑

i.e., $P\uparrow Q \Leftrightarrow \sim(P\wedge Q) \Leftrightarrow \sim P \vee \sim Q$

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

NOR: The word NOR is a combination of NOT & OR, where NOT Stands for Negation & OR for disjunction. The connective NOR is denoted by the symbol ↓
i.e., $P\downarrow Q \Leftrightarrow \sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

Basic properties of the connectives NAND and NOR

1) $P \uparrow Q = Q \uparrow P$ 2) $P \downarrow Q = Q \downarrow P$ (Commutative law)
2) The connectives ↑ & ↓ are not associative.

$P \uparrow Q \Leftrightarrow \Leftrightarrow \sim(P\wedge Q) \Leftrightarrow \sim P \vee \sim Q$

$\Leftrightarrow (\sim P \wedge T) \vee (\sim Q \wedge T)$

$\Leftrightarrow (\sim P \wedge (Q \vee \sim Q) \vee (\sim Q \wedge (P \vee \sim P)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (\sim Q \wedge P) \vee (\sim Q \wedge \sim P)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (\sim Q \wedge P)$

$P\downarrow Q \Leftrightarrow \sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

$\Leftrightarrow (\sim P \vee F) \wedge (\sim Q \vee F)$

$\Leftrightarrow (\sim P \vee (Q \wedge \sim Q)) \wedge (\sim Q \vee (P \wedge \sim P))$
$\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (\sim Q \vee P) \wedge (\sim Q \vee \sim P)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (\sim Q \vee P)$

<u>Exclusive OR:-</u> The Exclusive OR (or) XOR of P & Q is denoted by $P \oplus Q$ $(or) P \bar{\vee} Q$, is the proposition that is true when exactly one of P & Q is true, but not both and is false otherwise.

Truth Table for the Exclusive OR:

P	**Q**	$\boldsymbol{P \bar{\vee} Q}$
T	T	F
T	F	T
F	T	T
F	F	F

The Exclusive OR is also called the Exclusive Disjunction.

<u>Note:-</u> (i) $\boldsymbol{P} \bar{\vee} \boldsymbol{Q} \Leftrightarrow \boldsymbol{Q} \bar{\vee} P$ (Symmetric)
(ii) $(\boldsymbol{P} \bar{\vee} \boldsymbol{Q}) \bar{\vee} \mathrm{R} \Leftrightarrow \boldsymbol{P} \bar{\vee} (\boldsymbol{Q} \bar{\vee} \mathrm{R})$ (Associative)
(iii) $\boldsymbol{P} \wedge (\boldsymbol{Q} \bar{\vee} \mathrm{R}) \Leftrightarrow (\boldsymbol{P} \wedge Q) \bar{\vee} (\boldsymbol{P} \wedge R)$ (Distributive)

Normal Forms (canonical form)

We denote the word 'product' in place of 'conjunction' & 'sum' in place of 'Disjunction'.

* A Product of the variables & their negations in a formula is called as a 'elementary product'.

Eg:- $\sim P$, Q, P, $P \wedge Q$, $P \wedge \sim P$, $\sim P \wedge \sim Q$.

* A sum of the variables & their negations in a formula is called as the 'elementary sum'.

Eg:- P, Q, $P \vee Q$, $P \vee \sim Q$, $\sim P \vee \sim Q$, $\sim Q$.

Any Part of the elementary Product (sum) which is itself an elementary product (sum) is called as a factor.

Eg:- $P \wedge Q \wedge \sim P$

$\Leftrightarrow P \wedge \sim P \wedge Q$

$\because P \wedge \sim P$ is false, the whole elementary Product $P \wedge Q \wedge \sim P$ is false. $\therefore P \wedge \sim P$ is a factor of $P \wedge Q \wedge \sim P$.

$P \vee Q \vee \sim Q$

$\Leftrightarrow P \vee (Q \vee \sim Q)$

$\because (Q \vee \sim Q)$ is true, the whole elementary sum $P \vee Q \vee \sim Q$ is true. $\therefore (Q \vee \sim Q)$ is a factory of $P \vee Q \vee \sim Q$.

Eg:- Q, $Q \vee \sim P$, $\sim P \vee \sim Q$ are some of the factors of $Q \vee \sim P \vee \sim Q$.

$\sim Q$, $P \wedge \sim P$, $\sim Q \wedge P$ are some of the factors of $\sim Q \wedge P \wedge \sim P$.

* Disjunctive Normal form (d.n.f).

A formula which is equivalent to a given formula & which consists of a sum of elementary products is called a Disjunctive Normal form of the given formula.

Eg:- $(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge Q \wedge \sim Q)$.

This is a sum of elementary products.

$\therefore$ It is a Disjunctive Normal form.

Procedure to obtain a Disjunctive Normal form of a given formula:

1) If the connectives $\rightarrow$ & $\leftrightarrow$ are present in the given formula they are replaced by $\wedge, \vee$ & $\sim$

i.e $P \rightarrow Q$ is replaced by $\sim P \vee Q$ &

$P \leftrightarrow Q$ is replaced by either $(P \wedge Q) \vee (\sim P \wedge \sim Q)$

(or) $(\sim P \vee Q) \wedge (\sim Q \vee P)$.

2) If the negation is present before the given formula or a part of the given formula (not a variable). DeMorgan's laws are applied so that the negation is brought before the variables only.

3) Apply distributive laws in order to get the sum of elementary Products.

Note:-

1) The d.n.f of a given formula is not unique because different d.n.f can be obtained for a given formula if the distributive laws are applied in different ways.

3) A given formula is identically false if every elementary product appearing in its disjunctive normal form is identically false.

Obtain disjunctive normal form (d.n.f) of the following formulas:

1) $Q \rightarrow (Q \rightarrow P)$.

$\Leftrightarrow \sim Q \vee (Q \rightarrow P)$. [By Law of implication]

$\Leftrightarrow \sim Q \vee (\sim Q \vee P)$. ["]

$\Leftrightarrow (\sim Q \vee \sim Q) \vee P$ [By associative law).

$\Leftrightarrow \sim Q \vee P$. [By idempotent law $P \vee P = P$).

2) $(Q \rightarrow P) \wedge (\sim P \wedge Q)$.

$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \wedge Q)$ [By Law of implication).

$\Leftrightarrow (\sim Q \wedge \sim P \wedge Q) \vee (P \wedge \sim P \wedge Q)$. [Distributive law).

which is d.n.f of given formula.

3) $P \wedge (P \rightarrow Q)$.

$\Leftrightarrow P \wedge (\sim P \vee Q)$. [By law of implication]

$\Leftrightarrow (P \wedge \sim P) \vee (P \wedge Q)$ [By distributive law).

which is d.n.f of given formula.

4) $Q \wedge (P \vee \sim Q)$.

$\Leftrightarrow (Q \wedge P) \vee (Q \wedge \sim Q)$. [BY distributive law]

which is d.n.f of given formula.

5) $P \rightarrow ((P \rightarrow Q) \wedge \sim(\sim Q \vee \sim P))$.

$\Leftrightarrow \sim P \vee ((P \rightarrow Q) \wedge \sim(\sim Q \vee \sim P))$ [Law of implication]

$\Leftrightarrow \sim P \vee ((\sim P \vee Q) \wedge (Q \wedge P))$. ["]

$\Leftrightarrow \sim P \vee ((\sim P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P))$

$\Leftrightarrow \sim P \vee (\sim P \wedge P \wedge Q) \vee (Q \wedge P)$.

(By idempotent law $Q \wedge Q = Q$).

$\Leftrightarrow \sim P \vee (\sim P \wedge P \wedge Q) \vee (P \wedge Q)$

which is d.n.f of given formula.

6) $\sim(P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow (\sim(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \sim(P \wedge Q))$

[$\because P \leftrightarrow Q = (P \wedge Q) \vee (\sim P \wedge \sim Q)$]

$\Leftrightarrow (\sim P \wedge \sim Q) \wedge (P \wedge Q) \vee (P \vee Q) \wedge (\sim P \vee \sim Q)$.

[BY Demorgan's law]

$\Leftrightarrow$

$\Leftrightarrow (\sim P \wedge \sim Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \sim P)) \vee ((P \vee Q) \wedge \sim Q))$

[BY associative & distributive laws).

$\Leftrightarrow (\sim P \wedge \sim Q \wedge P \wedge Q) \vee (P \wedge \sim P) \vee (Q \wedge \sim P) \vee$
$(P \wedge \sim Q) \vee (Q \wedge \sim Q)$.

[By distributive law]

which is the required d.n.f.

* Conjunctive Normal Form (c.n.f).

A formula which is equivalent to a given formula ~~&~~ and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

(or)

suppose A, B are formulas which are equivalent & B contains the product of elementary sums then B is called conjunction normal form of A.

Note :-

1) The conjunctive normal form is not unique.

2) A given formula is ~~identiti~~ identically true (or) a tautology if every elementary sum in its conjunctive normal form is identically true.

Eg:- $P \wedge (Q \vee R)$.

which is already in the c.n.f.

The given formula can be written as.

$\Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ (By distributive law)

$\Leftrightarrow ((P \wedge Q) \vee P) \vee ((P \wedge Q) \vee R)$. (,,)

$\Leftrightarrow (P \vee P) \wedge (Q \vee P) \wedge (P \vee R) \wedge (Q \vee R)$.

which is another equivalent c.n.f.

$\therefore$ conjunctive normal form is not unique.

Obtain conjunctive normal forms of the following formulas:

1) $P \wedge (P \rightarrow Q)$.

$\Leftrightarrow P \wedge (\sim P \vee Q)$.

$\Leftrightarrow (P \vee P) \wedge (\sim P \vee Q)$. $[\because P \vee P = P]$

which is required c.n.f.

2) $\sim (P \vee Q) \leftrightarrow (P \wedge Q)$.

$\Leftrightarrow [\sim (P \vee Q) \rightarrow P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \sim (P \vee Q))$.

$[\because P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow R)]$.

$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\sim (P \wedge Q) \vee \sim (P \vee Q))$.

[By law of implication

i.e $P \rightarrow Q \Leftrightarrow \sim P \vee Q$]

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\sim P \vee \sim Q) \vee (\sim P \wedge \sim Q)$.

(By distributive law & Demorgan's law)

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\sim P \vee \sim Q \vee \sim P) \wedge (\sim P \vee \sim Q \vee \sim Q)$

(By distributive law).

which is in conjunctive normal form.

(or)

$\Leftrightarrow (P \vee Q) \wedge (P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (\sim P \vee \sim Q)$.

$(\because Q \vee Q = Q,\ P \vee P = P)$.

$(\sim Q \vee \sim Q = \sim Q)$

$\Leftrightarrow (P \vee Q) \wedge (\sim P \vee \sim Q) \quad [\because P \wedge P = P]$.

which is also required c.n.f.

3) $((P \to Q) \wedge \sim P) \to \sim Q$.

$\Leftrightarrow ((\sim P \vee Q) \wedge \sim P) \to \sim Q \quad (\because P \to Q \Leftrightarrow \sim P \vee Q)$.

Law of implication

$\Leftrightarrow \sim ((\sim P \vee Q) \wedge \sim P) \vee \sim Q \quad ($ " $)$

$\Leftrightarrow ((P \wedge \sim Q) \vee P) \vee \sim Q$ (By Demorgan's Law).

$\Leftrightarrow ((P \vee P) \wedge (\sim Q \vee P)) \vee \sim Q$. (By distributive Law).

$\Leftrightarrow (P \vee P \vee \sim Q) \wedge (\sim Q \vee P \vee \sim Q) \quad ($ " $)$.

$\Leftrightarrow (P \vee \sim Q) \wedge (\sim Q \vee P)$.

which is required c.n.f

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4) $((P \rightarrow Q) \wedge \sim Q) \rightarrow \sim P$

$\Leftrightarrow ((\sim P \vee Q) \wedge \sim Q) \rightarrow \sim P$. ($\because P \rightarrow Q \Leftrightarrow \sim P \vee Q$).

$\Leftrightarrow \sim((\sim P \vee Q) \wedge \sim Q) \vee \sim P$. (")

$\Leftrightarrow ((P \wedge \sim Q) \vee Q) \vee \sim P$. (By Demorgans Law).

$\Leftrightarrow ((P \vee Q) \wedge (\sim Q \vee Q)) \vee \sim P$ (By distributive Law).

$\Leftrightarrow (P \vee Q \vee \sim P) \wedge (\sim Q \vee Q \vee \sim P)$. (").

which is required c.n.f.

and the given formula is a tautology.

Show that the formula $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

First we obtain a conjunctive normal form of the given formula.

$Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

$\Leftrightarrow Q \vee (P \vee \sim P) \wedge (\sim Q \vee \sim Q)$.

$\Rightarrow (Q \vee (P \vee \sim P)) \wedge (\sim Q)$

$\Rightarrow (Q \vee P \vee \sim P) \wedge (Q \vee \sim Q)$.

Since each of the elementary sums is a tautology, the given formula is a tautology.

* A statement formula is said to be in the normal form (canonical form) if

1) only the three connections $\sim, \wedge, \vee$ have been used.
2) Distributive law has been applied.
3) Paranthesis has not been used for the same connective.

Eg:- $P \wedge (Q \wedge R)$ is $P \wedge Q \wedge R$.

* Min term

Min term consists of conjunction of variables or its negation but not both, appears only once.

[Eg:- $P \wedge Q$ or $Q \wedge P$ not both].

Note:-

1) For the variables P, Q there are 2^2 min terms
$P \wedge Q$, $P \wedge \sim Q$, $\sim P \wedge Q$ & $\sim P \wedge \sim Q$.
2) For the variables P, Q & R there are 2^3 Min terms.

$P \wedge Q \wedge R$, $P \wedge Q \wedge \sim R$, $P \wedge \sim Q \wedge R$, $P \wedge \sim Q \wedge \sim R$,

$\sim P \wedge Q \wedge R$, $\sim P \wedge \sim Q \wedge R$, $\sim P \wedge Q \wedge \sim R$, $\sim P \wedge \sim Q \wedge \sim R$.

Truth Table

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F	F
T	F	F	F	T	T	F	F
F	T	F	T	F	F	T	F
F	F	F	T	T	F	F	T

Note:- No two min terms are equivalent.

* Max term

Max term consists of disjunction of variables or its negations but not both, appears only once.

Note:-
For the variables P, Q there are 2^2 max terms

$P \vee Q$, $P \vee \sim Q$, $\sim P \vee Q$ & $\sim P \vee \sim Q$.

For the variables p.statements P, Q, R there are $2^3 = 8$ Max terms

$P \vee Q \vee R$, $P \vee Q \vee \sim R$, $P \vee \sim Q \vee R$, $\sim P \vee Q \vee R$, $P \vee Q \vee \sim R$,

$\sim P \vee Q \vee \sim R$, $\sim P \vee \sim Q \vee R$, $\sim P \vee \sim Q \vee \sim R$.

No two max terms are equivalent.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$P \vee \sim Q$	$\sim P \vee Q$	$\sim P \vee \sim Q$
T	T	F	F	T	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	F	T	T
F	F	T	T	F	~~F~~ T	T	T

Principal disjunctive normal form (P.d.n.f)

An equivalent formula consisting of disjunctions of min terms only is known as P.d.n.f.

There are two methods to obtain P.d.n.f of the given statement formula.

1) Truth table method.

2) Replacement method (without using truth table).

obtain principal disjunctive normal form (p.d.n.f) of each of the following:

i) $P \rightarrow Q$.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$\sim P \wedge \sim Q$	$P \rightarrow Q$
T	T	F	F	T	F	F	F	T
T	F	F	T	F	T	F	F	F
F	T	T	F	F	F	T	F	T
F	F	~~F~~ T	~~F~~ T	F	F	F	F	T

Min terms are $P \wedge Q$, $\sim P \wedge Q$, $\sim P \wedge \sim Q$

p.d.n.f of $P \rightarrow Q$ ~~are~~ is

$(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$.

obtain the p.d.n.f of the following formula.

$P \vee (\sim P \rightarrow (Q \vee (\sim Q \rightarrow R)))$.

P	Q	R	$\sim P$	$\sim Q$	$\sim Q \rightarrow R$	$\underbrace{Q \vee (\sim Q \rightarrow R)}_{A}$	$\underbrace{\sim P \rightarrow A}_{B}$	$P \vee B$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	T	F	T	T	T	T
T	F	F	F	T	F	F	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	F	F

Min terms are $P\wedge Q\wedge R$, $P\wedge Q\wedge \sim R$, $P\wedge \sim Q\wedge R$, $\sim P\wedge Q\wedge R$, $P\wedge \sim Q\wedge \sim R$, $\sim P\wedge Q\wedge \sim R$, $\sim P\wedge \sim Q\wedge R$.

P.d.n.f of $P\vee(\sim P\rightarrow(Q\vee(\sim Q\rightarrow R)))$ is

$(P\wedge Q\wedge R)\vee(P\wedge Q\wedge \sim R)\vee(P\wedge \sim Q\wedge R)\vee(\sim P\wedge Q\wedge R)\vee$
$(P\wedge \sim Q\wedge \sim R)\vee(\sim P\wedge Q\wedge \sim R)\vee(\sim P\wedge \sim Q\wedge R)$.

* Replacement method (i.e without using Truth tables)

1) Replace conditional & bi-conditional by their equivalent formula containing only $\wedge$, $\vee$ & $\sim$.

2) use Demorgan's laws & distributive laws.

3) Any elementary product which is a contradiction can be dropped ($\because$ $P\vee F = P$)

4) If identical min terms appear in the disjunction then the repeated one is ignored (or) dropped.

Note:-

1) $P\vee F = P$ 2) $P\wedge T = P$ | $P\vee T = T$, $P\wedge F = F$ } domination Laws.

3) $P\vee \sim P = T$ 4) $P\wedge \sim P = F$

① & ② are identity laws

③ & ④ ~~are~~

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obtain P.d.n.f for following formulas without using truth-tables.

1) $P \leftrightarrow Q$.

$\Leftrightarrow (P \to Q) \wedge (Q \to P)$.

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim Q \vee P)$ $[\because P \to Q \Leftrightarrow \sim P \vee Q$ Law of implication$]$

$\Leftrightarrow (\sim P \wedge \sim Q) \vee (\sim P \wedge P) \vee (Q \wedge \sim Q) \vee (Q \wedge P)$.

$\Leftrightarrow (\sim P \wedge \sim Q) \vee (Q \wedge P)$ $(\because \sim P \wedge P = F, Q \wedge \sim Q = F)$.

which is a P.d.n.f.

2) $\sim P \vee Q$.

$\Leftrightarrow (\sim P \wedge T) \vee (T \wedge Q)$. $(\because P \wedge T = P)$

$\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee ((P \vee \sim P) \wedge Q)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q) \vee (\sim P \wedge Q)$.

(By distributive law).

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q)$.

$(\because P \vee P = P$, commutative law$)$.

$\Leftrightarrow (P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

which is a P.d.n.f.

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3) $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow ((P \wedge Q) \wedge T) \vee ((\sim P \wedge R) \wedge T) \vee ((Q \wedge R) \wedge T)$.

$\Leftrightarrow ((P \wedge Q) \wedge (R \vee \sim R)) \vee ((\sim P \wedge R) \wedge (Q \vee \sim Q)) \vee$
$((Q \wedge R) \wedge (P \vee \sim P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q)$
$\vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$.

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee$
$(\sim P \wedge \sim Q \wedge R)$.

$(\because P \vee P = P)$.

which is required p.d.n.f.

4) $P \to ((P \to Q) \wedge \sim(\sim Q \vee \sim P))$

$\Leftrightarrow \sim P \vee ((P \to Q) \wedge \sim(\sim Q \vee \sim P))$ $(\because P \to Q \Leftrightarrow \sim P \vee Q)$.

$\Leftrightarrow \sim P \vee ((\sim P \vee Q) \wedge (Q \wedge P))$ (By Law of implication & Demorgan's law).

$\Leftrightarrow \sim P \vee (\sim P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)$ (By distributive law).

$\Leftrightarrow \sim P \vee ((\sim P \wedge P) \wedge Q) \vee (Q \wedge P)$. (By associative law & $Q \wedge Q \Leftrightarrow Q$).

$\Leftrightarrow \sim P \vee (F \wedge Q) \vee (P \wedge Q)$ ($\because$ By $\sim P \wedge P \Leftrightarrow F$ & commutative law).

$\Leftrightarrow \sim P \vee F \vee (P \wedge Q)$. $(\because F \wedge Q = F)$.

$\Leftrightarrow \sim P \vee (P \wedge Q)$ $(\because P \vee F = P)$.

$\Leftrightarrow (\neg P \wedge T) \vee (P \wedge Q)$ $(\because P \wedge T = P)$.

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$ $(\because P \vee \neg P \Leftrightarrow T)$,

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$. (By distributive law).

which is in required P.d.n.f.

Principal conjunctive Normal Form:
(Product of sums canonical form).

An equivalent formula consisting of conjunctions of the max terms only is known as its principal conjunctive Normal form (P.c.n.f).

The method for obtaining the P.c.n.f for a given formula is similar to the method for the P.d.n.f.

Method to obtain P.c.n.f of a given formula.

If the principal disjunctive (conjunctive) normal form of a given formula containing 'n' variables is known, then the principal disjunctive (conjunctive) normal form of the $\neg A$ will consist of the disjunction (conjunction) of the remaining min terms (max terms) which do not appear in the principal disjunctive (conjunctive) normal form of A. From $A \Leftrightarrow \neg(\neg A)$ one can obtain the principal conjunctive (disjunctive) normal form of A by repeated applications of Demorgan's laws to the principal disjunctive (conjunctive) normal form of $(\neg A)$

$\Leftrightarrow (\sim P \wedge T) \vee (P \wedge Q)$ $(\because P \wedge T = P)$.

$\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge Q)$ $(\because P \vee \sim P \Leftrightarrow T)$,

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q)$. (By distributive law).

which is in required P.d.n.f.

Principal conjunctive Normal Form:
(Product of sums canonical form).

An equivalent formula consisting of conjunctions of the max terms only is known as its principal conjunctive Normal form (P.C.n.f).

The method for obtaining the P.C.n.f for a given formula is similar to the method for the P.d.n.f.

Method to obtain P.c.n.f of a given formula.

If the principal disjunctive (conjunctive) normal form of a given formula containing 'n' variables is known, then the principal disjunctive (conjunctive) normal form of the $\sim A$ will consist of the disjunction (conjunction) of the remaining min terms (max terms) which do not appear in the principal disjunctive (conjunctive) normal form of A. From $A \Leftrightarrow \sim(\sim A)$ one can obtain the principal conjunctive (disjunctive) normal form of A by repeated applications of Demorgan's laws to the principal disjunctive (conjunctive) normal form of $(\sim A)$

Obtain the principal conjunctive normal form of the following formula:

1) $\sim (P \vee Q)$.

Let $A : \sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ which is in P.d.n.f

P.d.n.f of $\sim A : (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$. (Remaining terms)

P.c.n.f of $\sim(\sim A)$: $(\sim P \vee \sim Q) \wedge (\sim P \vee Q) \wedge (P \vee \sim Q)$

which is P.c.n.f of A $(\because \sim(\sim A) = A)$.

2) $\sim (P \rightarrow Q)$.

$\Leftrightarrow \sim (\sim P \vee Q)$.

$\Leftrightarrow P \wedge \sim Q$ which is P.d.n.f.

P.d.n.f of $\sim A = (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$ (Remaining terms).

P.c.n.f of $\sim(\sim A) = (\sim P \vee \sim Q) \wedge (\sim P \vee Q) \wedge (P \vee \sim Q)$.

which is P.c.n.f of A.

3) $(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$

Let $A = (P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$.

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$.

$\Leftrightarrow (P \wedge (Q \vee \sim Q)) \vee (\sim P \wedge Q)$.

$\Leftrightarrow (P \wedge T) \vee (\sim P \wedge Q)$.

$\Leftrightarrow P \vee (\sim P \wedge Q)$.

$\Leftrightarrow (P \vee \sim P) \wedge (P \vee Q)$.

$\Leftrightarrow T \wedge (P \vee Q)$. $[\because P \vee \sim P = T]$

$\Leftrightarrow P \vee Q$. $(\because T \wedge P = P)$.

(or)

Let $A = (P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$.

$\sim A = (\sim P \wedge \sim Q)$ · p.d.n.f of $\sim A$.

p.c.n.f of $\sim(\sim A) = P \vee Q$ which is p.c.n.f of A.

4) $(P \wedge Q) \vee (\sim P \wedge Q \wedge R)$

Let $A = (P \wedge Q) \vee (\sim P \wedge Q \wedge R)$.

$\Leftrightarrow A = ((P \wedge Q) \wedge T) \vee (\sim P \wedge Q \wedge R)$.

$\Leftrightarrow A = ((P \wedge Q) \wedge (R \vee \sim R)) \vee (\sim P \wedge Q \wedge R)$.

$\Leftrightarrow A = (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R)$.

which is p.d.n.f of A.

p.d.n.f of $\sim A = (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee$ $(\sim P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge \sim R)$ (Remaining terms)

$\sim(\sim A) = (\sim P \vee Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge (P \vee Q \vee \sim R) \wedge$ $(P \vee \sim Q \vee R) \wedge (P \vee Q \vee R)$.

which is p.c.n.f of A.

5) $(P \wedge Q \wedge R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge \sim Q \wedge \sim R)$

Let $A = (P \wedge Q \wedge R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge \sim Q \wedge \sim R)$ which is in p.d.n.f.

$\sim A = (P \wedge \sim Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R)$. (Remaining terms of A).

$\sim(\sim A) = \sim[(P \wedge \sim Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R)]$.

$\Leftrightarrow A = \sim(P \wedge \sim Q \wedge R) \wedge \sim(P \wedge Q \wedge \sim R) \wedge \sim(P \wedge \sim Q \wedge \sim R) \wedge \sim(\sim P \wedge \sim Q \wedge R) \wedge \sim(\sim P \wedge Q \wedge \sim R)$

$\Leftrightarrow A = (\sim P \vee Q \vee \sim R) \wedge (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee Q \vee R) \wedge (P \vee Q \vee \sim R) \wedge (P \vee \sim Q \vee R)$.

which is in p.c.n.f (Product of sums canonical form).

6) Obtain the principal conjunctive normal form of the formula A given by $(\sim P \rightarrow R) \wedge (Q \leftrightarrow P)$ & hence find disjunctive normal form of A.

Given $A = (\sim P \rightarrow R) \wedge (Q \leftrightarrow P)$.

$\Leftrightarrow A = (P \vee R) \wedge (Q \rightarrow P) \wedge (P \rightarrow Q)$.

$[\because P \rightarrow Q \Leftrightarrow \sim P \vee Q$ &
$Q \leftrightarrow P \Leftrightarrow (Q \rightarrow P) \wedge (P \rightarrow Q)]$.

$\Leftrightarrow A = (P \vee R) \wedge (\sim Q \vee P) \wedge (\sim P \vee Q)$.

$\Leftrightarrow A = ((P \vee R) \vee F) \wedge ((\sim Q \vee P) \vee F) \wedge ((\sim P \vee Q) \vee F)$

$\Leftrightarrow A = ((P \vee R) \vee (Q \wedge \sim Q)) \wedge ((\sim Q \vee P) \vee (R \wedge \sim R)) \wedge$
$((\sim P \vee Q) \vee (R \wedge \sim R))$

$\Leftrightarrow A = (P \vee R \vee Q) \wedge (P \vee R \vee \sim Q) \wedge (\sim Q \vee P \vee R) \wedge$
$(\sim Q \vee P \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$.

$\Leftrightarrow A = (P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R) \wedge$
$(\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$.

which is in P.C.N.F.

$\sim A = (P \vee Q \vee \sim R) \wedge (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee \sim Q \vee \sim R)$.

(Remaining max terms) is P.C.N.F of $\sim A$.

$\sim(\sim A) = \sim((P \vee Q \vee \sim R) \wedge (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee \sim Q \vee \sim R))$

$A = \sim(P \vee Q \vee \sim R) \vee \sim(\sim P \vee \sim Q \vee R) \vee \sim(\sim P \vee \sim Q \vee \sim R).$

$= (\sim P \wedge \sim Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (P \wedge Q \wedge R).$

(By Demorgan's Law

$\sim(P \wedge Q) = \sim P \vee \sim Q$).

which is P.d.n.f of A.

Show that the following are equivalent formulas.

1) $P \vee (P \wedge Q) \Leftrightarrow P$.

Let us consider

$P \vee (P \wedge Q) \Leftrightarrow (P \wedge T) \vee (P \wedge Q)$ ($\because P \wedge T = P$)

$\Leftrightarrow (P \wedge (Q \vee \sim Q)) \vee (P \wedge Q)$.

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee P \wedge Q)$. (Distributive law).

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q)$ ($\because P \vee P = P$). $\rightarrow$ 1.

$P \Leftrightarrow P \wedge T$

$\Leftrightarrow P \wedge (Q \vee \sim Q)$.

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \rightarrow$ 2.

$\because$ 1 & 2 are ~~equivalent~~ equal

$\therefore$ The given formulas are equivalent.

2) $P \vee (\sim P \wedge Q) \Leftrightarrow P \vee Q$.

Let us consider $P \vee (\sim P \wedge Q)$.

$\Leftrightarrow (P \wedge T) \vee (\sim P \wedge Q)$. ($\because P \wedge T = P$).

$\Leftrightarrow (P \wedge (Q \vee \sim Q)) \vee (\sim P \wedge Q)$. ($\because Q \vee \sim Q = T$).

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$. $\rightarrow$ 1

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$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T).$

$\Leftrightarrow (P \wedge (Q \vee \sim Q)) \vee (Q \wedge (P \vee \sim P)).$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (Q \wedge P) \vee (Q \wedge \sim P)$

(By distributive law).

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q) \vee (Q \wedge \sim P). \rightarrow (2)$

$(\because P \vee P = P).$

$\because$ (1) & (2) are equal.

$\therefore$ The given formulas are equivalent.

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Show that $\{\uparrow\}$, $\{\downarrow\}$ are functionally complete.

Proof:-

In order to prove, it is sufficient to show that the sets of connectives $\{\wedge, \vee\}$ & $\{\vee, \sim\}$ can be expressed either interms of $\uparrow$ alone or interms of $\downarrow$ alone.

i) To S.T $\{\vee, \sim\}$ is functionally complete, it is enough to S.T $\sim$ & $\vee$ can be expressed interms of $\downarrow$ alone

$$\sim P \Leftrightarrow \sim P \wedge \sim P$$
$$\Leftrightarrow \sim (P \vee P)$$
$$\Leftrightarrow P \downarrow P$$

& $P \vee Q \Leftrightarrow \sim(\sim(P \vee Q))$

$$\Leftrightarrow \sim(\sim(P \vee Q) \vee \sim(P \vee Q))$$
$$\cancel{\Leftrightarrow \sim(P \downarrow Q) \downarrow \sim(P \vee Q)}$$
$$\Leftrightarrow \sim((P \downarrow Q) \vee (P \downarrow Q))$$
$$\Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q).$$

Then $\{\downarrow\}$ is functionally complete.

ii) To S.T $\{\wedge, \sim\}$ is functionally complete, it is enough to S.T $\sim$ & $\wedge$ can be expressed in terms of $\uparrow$ alone.

$\sim P \Leftrightarrow \sim P \vee \sim P$

$\Leftrightarrow \sim(P \wedge P)$

$\Leftrightarrow P \uparrow P$

$\forall$ $P \wedge Q \Leftrightarrow \sim(\sim(P \wedge Q))$

$\Leftrightarrow \sim(\sim(P \wedge Q) \wedge \sim(P \wedge Q))$

$\Leftrightarrow \sim((P \uparrow Q) \wedge (P \uparrow Q))$

$\Leftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$.

Then $\{\uparrow\}$ is a functionally complete set.

The following equivalences express $\sim$, $\wedge$ & $\vee$ interms of $\uparrow$ alone.

$$\sim P \Leftrightarrow \sim P \vee \sim P$$
$$\Leftrightarrow \sim (P \wedge P)$$
$$\Leftrightarrow P \uparrow P$$

$$P \vee Q \Leftrightarrow \sim (\sim P \wedge \sim Q)$$
$$\Leftrightarrow \sim P \uparrow \sim Q$$
$$\Leftrightarrow (P \uparrow P) \uparrow (Q \uparrow Q).$$

$$P \wedge Q \Leftrightarrow \sim (\sim (P \wedge Q))$$
$$\Leftrightarrow \sim (\sim (P \wedge Q) \wedge \sim (P \wedge Q))$$
$$\Leftrightarrow \sim (P \wedge Q) \uparrow \sim (P \wedge Q)$$
$$\Leftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q).$$

In a similar manner, the following equivalences express $\sim$, $\vee$ & $\wedge$ interms of $\downarrow$ alone.

$$\sim P \Leftrightarrow \sim P \wedge \sim P$$
$$\Leftrightarrow \sim (P \vee P)$$
$$\Leftrightarrow P \downarrow P$$

$P \vee Q \Leftrightarrow \sim(\sim(P \vee Q))$

$\Leftrightarrow \sim(\sim(P \vee Q) \vee \sim(P \vee Q))$

$\Leftrightarrow \sim(P \vee Q) \downarrow \sim(P \vee Q)$

$\Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$

$P \wedge Q \Leftrightarrow \sim(\sim P \vee \sim Q)$

$\Leftrightarrow \sim P \downarrow \sim Q$.

$\Leftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$.

Express $P \rightarrow (\sim P \rightarrow Q)$ in terms of $\uparrow$ (NAND) only.

Express the same formula in terms of $\downarrow$ (NOR) only.

$P \rightarrow (\sim P \rightarrow Q)$

$\Leftrightarrow P \rightarrow (P \vee Q)$

$\Leftrightarrow \sim P \vee (P \vee Q)$

$\Leftrightarrow (\sim P \vee P) \vee Q$

$\Leftrightarrow T \vee Q$

$\Leftrightarrow T \quad \Leftrightarrow \sim P \vee P$.

$\Leftrightarrow \sim(P \wedge \sim P)$

$\Leftrightarrow P \uparrow \sim P$.

$\Leftrightarrow P \uparrow (P \uparrow P)$

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$P \to (\sim P \to Q)$

$\Leftrightarrow P \to (P \vee Q)$

$\Leftrightarrow \sim P \vee (P \vee Q)$

$\Leftrightarrow (\sim P \vee P) \vee Q$

$\Leftrightarrow T \vee Q$

$\Leftrightarrow T$

$\Leftrightarrow P \vee \sim P$.

$\Leftrightarrow \sim [\sim (P \vee \sim P)]$

$\Leftrightarrow \sim [\sim (P \vee \sim P) \vee \sim (P \vee \sim P)]$

$\Leftrightarrow \sim (P \vee \sim P) \downarrow \sim (P \vee \sim P)$.

$\Leftrightarrow (P \downarrow \sim P) \downarrow (P \downarrow \sim P)$

$\Leftrightarrow P \downarrow (P \downarrow P) \downarrow P \downarrow (P \downarrow P)$ //.

$P \uparrow Q \Leftrightarrow \sim (P \wedge Q)$.

$\Leftrightarrow \sim P \vee \sim Q$

$\Leftrightarrow (\sim P \wedge T) \vee (\sim Q \wedge T)$

$\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (\sim Q \wedge (P \vee \sim P))$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (\sim Q \wedge P) \vee (\sim Q \vee \sim P)$

$\Leftrightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (\sim Q \wedge P)$

$$
\begin{aligned}
P \overline{\vee} Q &\Leftrightarrow \sim(P \vee Q) \\
&\Leftrightarrow \sim P \wedge \sim Q \\
&\Leftrightarrow (\sim P \vee F) \wedge (\sim Q \vee F) \\
&\Leftrightarrow (\sim P \vee (Q \wedge \sim Q)) \wedge (\sim Q \vee (P \vee \sim P)) \\
&\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (\sim Q \vee P) \wedge (\sim Q \vee \sim P) \\
&\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (\sim Q \vee P)
\end{aligned}
$$