5.1

Graph Theory (UNIT-V)

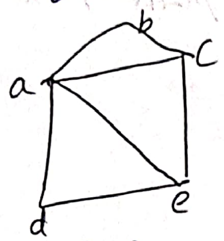
DM

Planar Graphs:

Definition: A Graph 'G' is said to be planar if it can be drawn in a plane without crossing its edges except at common vertices.

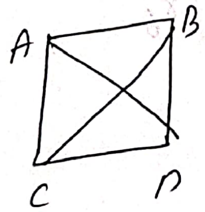
Otherwise it is called non-planar graph.

Ex:

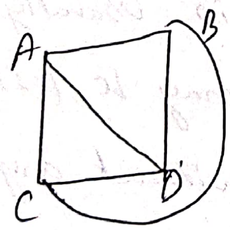


It is a planar graph.

In this graph no edges cross one another except at vertices. Hence it is a planar graph.



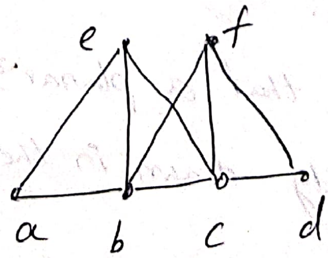
non planar graph



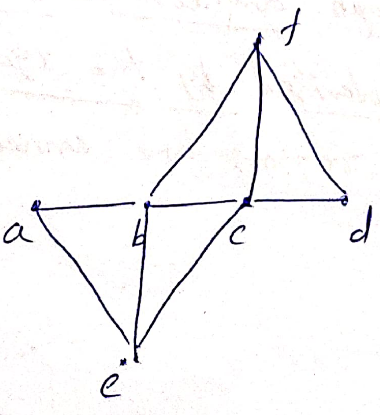
planar graph.

Example: Draw the planar graph for the following graph.

non-planar graph



Sol:



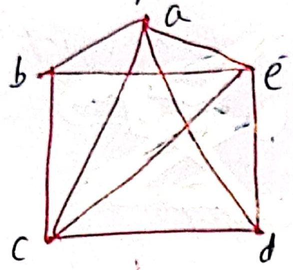
⇒ planar graph.

Regions of the planar graph

A region of the planar graph is defined to be an area of the plane that is bounded by edges and is not further divided into sub areas.

Ex: planar representation of the graph.

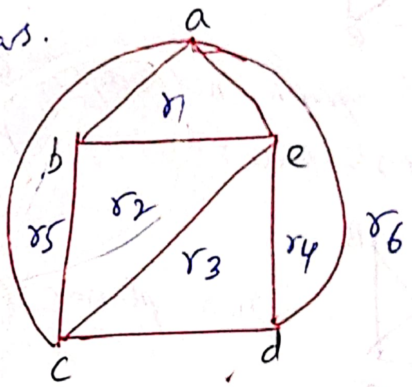
G:



Divides the plane into ~~five~~ "6" regions.

It can be drawn as.

Planar graph:



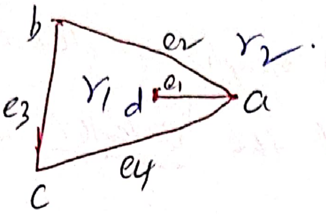
⇒ A graph may be planar even if it is usually drawn with edge crossings, since it may be possible to draw it in a different way without any edge crossings.

⇒ we say that a planar graph is a plane graph if it is already drawn in the plane without edge crossings

Note: In the above graph divides the plane into regions

→ A region is characterized by the cycle that forms its boundary. these regions are connected portions of the plane.

Ex) Find the degree of the region for the following graph.



sol:- boundary of r_1 — d - a - b - c - a - d.

the boundary of r_1 includes the edge "e_1" twice.

boundary of r_2 — a - b - c - a.

degree of r_1 — 5

degree of r_2 — 3.

* Euler's formula for the planar graphs:

If 'G' is a connected planar graph, then any drawing of 'G' in the plane as a plane graph will always form,

$\boxed{|R| = |E| - |V| + 2}$ regions or.

$\boxed{|R| - |E| + |V| = 2}$ regions. including exterior region.

Where $|R|$ = no: of regions.

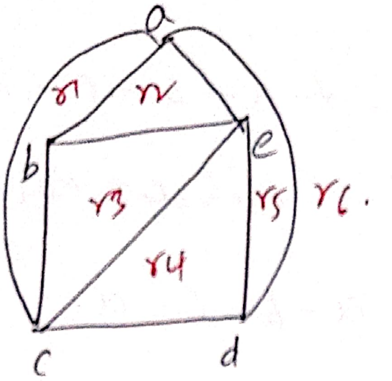
$|E|$ = no: of edges.

$|V|$ = no: of vertices.

Exterior Region:-

In each plane, A Graph 'G' determines the region of infinite area called the "exterior region" of G.

Ex:



In the above planar graph "r_6" is the "exterior" region.

⇒ The vertices and edges of 'G' incident with a region "r" make up the boundary of the region 'r'.

Ex: boundary of the region r_1 — a-b-c-a

" r_2 — a-b-e-a

" r_3 — b-c-e-b

" r_4 — c-d-e-c

" r_5 — a-e-d-a

" r_6 — a-c-d-a.

Degree of the region:

→ The degree of the region 'r' is the length of its boundary, denoted by deg(r).

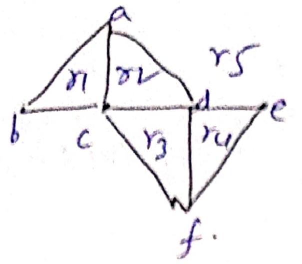
Ex:- $\deg(r_4) = 3$ $\deg(r_1) = 3$.

$\deg(r_6) = 3$.   ... etc.

$\deg(r_2) = 3$

$\deg(r_3) = 3$.

Example: Find the no: of regions for the following graph.



Sol: In the above planar graph.

Given that

no: of vertices $|V| = 6$.

no: of Edges $|E| = 9$.

no: of regions $|R| = 5$.

By Euler's formula we have.

$$|R| - |E| + |V| = 2$$

$$5 - 9 + 6 = 2$$

$$11 - 9 = 2$$

$$2 = 2.$$

Ex: Show that in a connected planar linear graph with 6 vertices and 12 edges, each of the region is bounded by 3 edges.

Sol: let $|E|$ and $|V|$ be the no: of edges and no: of vertices in the graph 'G'.

given that $|E| = 12$

$|V| = 6$.

let 'R' be the no: of regions of the graph 'G'.

By Eulers formula, we have

$$|R| = -|V| + |E| + 2 \quad (\because |E| - |V| + 2)$$

$$= -6 + 12 + 2$$

$$= 8.$$

$\therefore |R| = 8.$

And also we know that.

total degree of the regions in a planar graph

is $\sum_{i=1}^{|R|} d(r_i) = 2|E|$

$d(r_i)$ = degree of the region

$$\sum_{i=1}^{|R|} d(r_i) = 2(12) = 24.$$

$\because$ each region is bounded by 3 edges.

$\therefore 3 \times 8 = 24.$ = total degree of the regions in a graph.

$$\therefore \sum_{i=1}^{|R|} d(r_i) = 2|E| = 24.$$

Hence proved.

Ex: Suppose that connected planar graph has 20 vertices. each of degree 3. How many regions does a representation of this planar graph split into the plane.

Sol: Given that $|V| = 20$

degree of each ~~region~~ vertex is $= 3$.

total degree of vertices in a planar graph is

$$\Sigma \deg(v) \Rightarrow$$

$$= 3 \times 20 = 60.$$

we have $\Sigma \deg(V) = 2E$

$$60 = 2E$$

$$|E| = 60/2 = 30.$$

now By Euler's formula

$$|R| = |E| - |V| + 2$$

$$= 30 - 20 + 2$$

$$= 12.$$

$\therefore$ The regions are 12.

Ex: If 'G' is a polyhedral graph with 12 vertices and 30 edges. prove that degree of each region is 3.

Sol: Given that $|E| = 30$

$$|V| = 12.$$

By Euler's formula

$$|R| = |E| - |V| + 2$$

$$= 30 - 12 + 2$$

$$= 20.$$

When each region is r has a degree K.

$$\sum_{r \in R(G)} \deg(r) = 2|E|.$$

$$K \cdot R = 2|E|$$

$$K \cdot 20 = 2 \times 30$$

$$20K = 60$$

$$K = 60/20 = 3.$$

Each region has degree — 3.

Hence proved.

Graph Coloring or (Coloring of graph)

Definition:- let $G = (V, E)$ be graph with no multiple edges and $C = \{c_1, c_2, \dots c_x\}$ be any set of X colors. Any function $f: V \to C$ which assigns each vertex or edge to one color is called the "coloring of the graph 'G'". (or) 'Graph Coloring' (adjacent vertices (or) adjacent edges have different color).

Proper Coloring :-

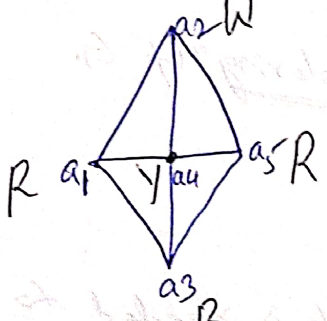
A Coloring is Called proper coloring if any two adjacent vertices have different Colors.

Ex: 4 - Coloring or 3 - coloring of graph.

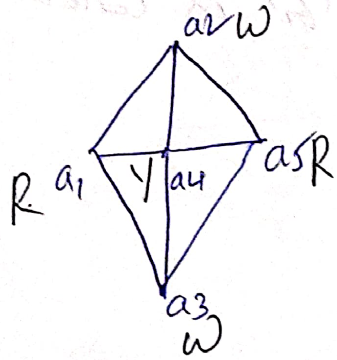
where $C = \{$Red, white, Blue, yellow$\}$

$= \{R, W, B, Y\}$.

Graph G ; The Same graph properly colored in two ways.



"4 - Colors"



"3 - Colors"

n - Coloring:

An n - coloring of graph 'G' is a Coloring of graph "G" using n - Colors. If G has no coloring then 'G' is said to be "n - Colorable".

Vertex Coloring:

The assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors is called "Vertex Coloring."

Edge Coloring:

Assignment of colors to the edges of graph G, so that no two adjacent edges receive the same color is called as "edge coloring" of G.

Chromatic number of G:

Def: The smallest no: of colors needed to produce a proper coloring of graph 'G' is called the chromatic no: of G.

And denoted by $X(G)$.

Welch-Powell Algorithm

Welch-Powell provides the algorithm for coloring of graph 'G'.

Steps of the Alg - (Welch-powell)

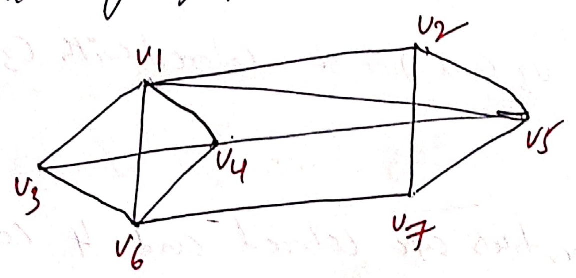
Step1: Order the verticals in G is decreasing degree.

step2: Use one Color for the first Vertex and color them in sequential Order each Vertex on the list, that is not adjacent to a Vertex previously Colored with this Color.

step3: Start again from the top of the list and repeat step2 using second Color.

step4: Repeat step3 untill all verticals are Colored.

Example: Find the chromatic no. of (G) i-e $X(G)$ for the following graph.



Sol: $\deg(V_1) = 5$

$\deg(V_2) = 3$

$\deg(V_3) = 3$

$\deg(V_4) = 4$

$\deg(V_5) = 4$

$\deg(V_6) = 4$

$\deg(V_7) = 3.$

Vertex:	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree:	5	4	4	4	3	3	3
Color:	C_1	C_2	C_3	C_3	C_2	C_4	C_1.

$\Rightarrow$ Color v_1 with C_1 and since v_7 is not a adjacent with v_1, color v_7 also with C_1.

$\Rightarrow$ Color v_4 with C_2, v_2 and v_7 are not adjacent with v_4. Since v_7 is already Colored.

Color v_2 with C_2.

$\Rightarrow$ Color v_5 with C_3. Since v_6 is not adjacent with v_5. Color v_6 with C_3. v_3 is also not adjacent with v_5 but it is adjacent with v_6.

Hence v_3 can not be Colored with C_3, Color v_3 with C_4.

$\Rightarrow$ All the vertices are colored and 4 colors are used.

Hence $\chi(G) \leq 4$.

But the vertices v_1, v_3, v_4 & v_6 are adjacent to each other. Hence at least 4 colors are required to color these vertices.

Thus the chromatic no. of G

$\chi(G) = 4$.

Ex:- Determine the chromatic no: of complete graph K_6, K_{10}, ~~and~~ and in general K_n graph.

Sol:- K_6 = six colors are needed to color K_6 graph.

Since in complete graph, every vertex is adjacent to every other vertex. And therefore for every vertex a different colors are needed.

$$\boxed{C(K_6) = 6.}$$

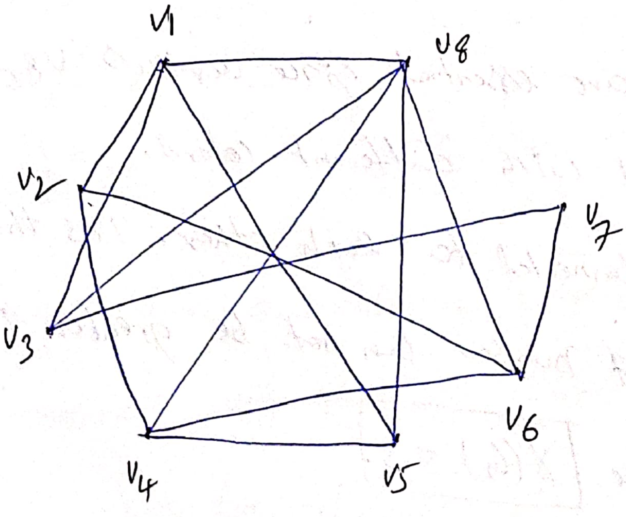
And also ten colors are needed to color the graph K_{10}.

$$\boxed{C(K_{10}) = 10.}$$

And 'n' colors are needed to color the graph K_n.

$$\boxed{C(K_n) = n.}$$

Ex:- Use the Welch-Powell algorithm to color the following graphs & find the chromatic number of ('n') ~~of~~ graph.



Sol: given that

$deg(v_1) = 4$

$deg(v_2) = 3$

$deg(v_3) = 3$

$deg(v_4) = 4$

$deg(v_5) = 3$

$deg(v_6) = 4$

$deg(v_7) = 2$

$deg(v_8) = 5$.

The sequences of vertices according to ~~deg~~ decreasing order of degrees in.

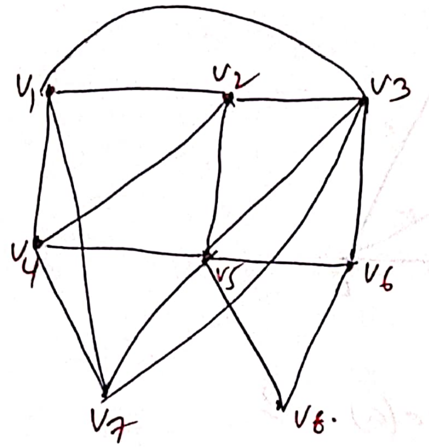
Degree:	5	4	4	4	3	3	3	2
Vertex:	v_8	v_1	v_4	v_6	v_2	v_3	v_5	v_7
Color:	C_1	C_2	C_2	C_3	C_1	C_3	C_3	C_1.

Three colors are essential since vertices v_8, v_4 & v_5 must be colored with different colors.

These are connected to each other. Thus the chromatic ~~no:ed~~ number can not be greater than 3.

Hence $\boxed{\chi(G) = 3.}$

Example: find the chromatic no. of graph G for the following graph.



Sol: The sequence of vertices according to decreasing order of degree is.

Degree:-	6	5	4	4	4	4	3	2
Vertices:-	v_5	v_3	v_1	v_4	v_7	v_2	v_6	v_8.
Color:	c_1	c_2	c_1	c_2	c_3	c_3	c_3	c_2.

$deg(v_1) = 4$ $\quad deg(v_4) = 4$ $\quad deg(v_7) = 4$

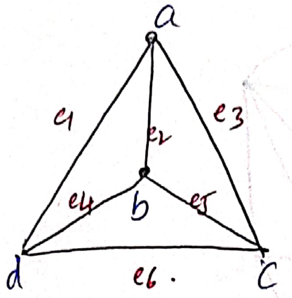
$deg(v_2) = 4$ $\quad deg(v_5) = 6$ $\quad deg(v_8) = 2$.

$deg(v_3) = 5$ $\quad deg(v_6) = 3$

Three colors are essential. since v_1, v_2 and v_3 must be colored with different colors.

Hence $\boxed{\chi(G) = 3}$

Example: Give the edge chromatic number for the following graph.



Sol:

degree(a) = 3

deg(b) = 3

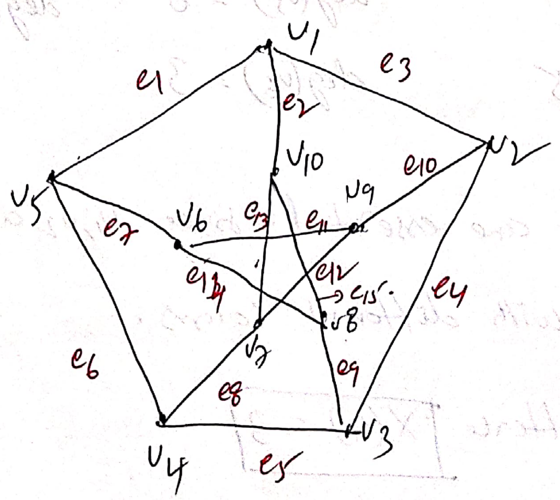
deg(c) = 3

deg(d) = 3.

Edge:	e_1	e_2	e_3	e_4	e_5	e_6
Color:	c_1	c_2	c_3	c_3	1	2

Edge Chromatic number ~~of~~ $X(G)$ is = 3.

Ex: Edge Chromatic number of the following graph.



Sol: This graph has 15-edges and 10 vertices.

$\Delta(G) = 3.$

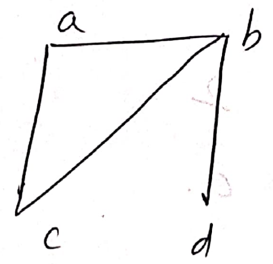
Edge :-	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
Color :	c_1	c_2	c_3	c_2	c_3	c_2	c_3	c_4	c_1	c_1	c_2	c_3	c_3	c_4	c_3

Edge Chromatic number is 4.

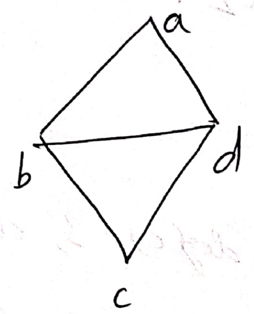
$\boxed{\therefore X(G) = 4.}$

Ex: find the Chromatic number of the given graph.

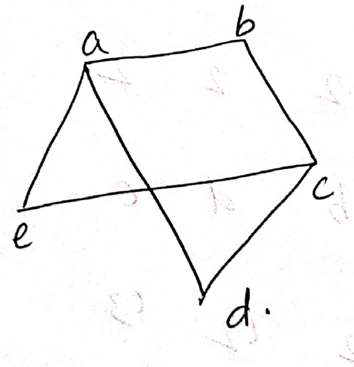
(a)



(b)



(c)



Sol: ① $deg(a) = 2$

$deg(b) = 3$

$deg(c) = 2$

$deg(d) = 1$

degree:	3	2	2	1
vertices:	b	a	c	d.
Color:	c_1	c_2	c_3	c_2.

The chromatic number $\chi(G) = 3$.

② $deg(a) = 2$ $deg(b) = 3$ $deg(c) = 2$ $deg(d) = 3$

degree:	3	3	2	2
vertices:	b	d	a	c.
Color:	c_1	c_2	c_3	c_2.

$\chi(G) = 3$.

③ $deg(a) = 3$, $deg(b) = 2$, $deg(c) = 3$, $deg(d) = 2$

$deg(e) = 2$.

degree:	3	3	2	2	2
vertices:	a	c	b	d	e.
Color:	c_1	c_1	c_2	c_2	c_3

$\chi(G) = 3$.

Euler Circuits - Hamiltonian Graphs :

Euler or Eulerian graphs

Eulerian path: A path of the graph 'G' is called an "Eulerian path", if it includes each edge of G exactly once, but vertices are may repeated.

Eulerian circuit:

A circuit of the graph 'G' is called an "Eulerian circuit" if it includes each of edge G exactly once.

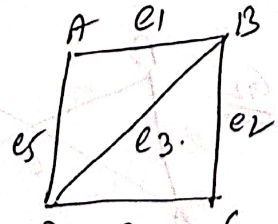
Eulerian graph:

A graph containing an Eulerian circuit is called an "Eulerian graph".

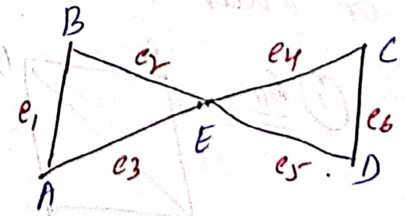
Examples:

Eulerian Graph Ex:-

① G_1



② G_2



→ A Graph 'G' containing Eulerian path between "B & D"

namely $B \xrightarrow{e_3} D \xrightarrow{e_4} C \xrightarrow{e_2} B \xrightarrow{e_1} A \xrightarrow{e_5} D$.

→ A Graph G_2 contains an Eulerian circuit namely.

$$A \xrightarrow{e_3} E \xrightarrow{e_4} C \xrightarrow{e_6} D \xrightarrow{e_5} E \xrightarrow{e_2} B \xrightarrow{e_1} A.$$

Since it includes each of edge exactly once.

∴ G_2 is an Euler or "Eulerian graph" because it is a "Euler circuit".

⇒ The necessary and sufficient conditions for the existence of Euler circuit and Euler path given in two theorems without proof.

Theorem-1:

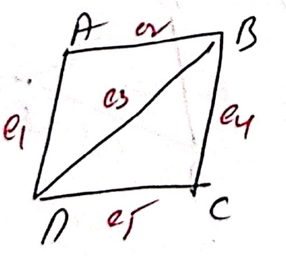
A connected graph contains an "Euler circuit", if and only if each of its vertices is "even degree".

Theorem-2:

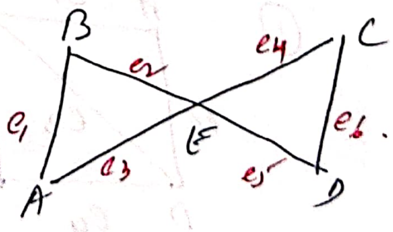
A connected graph contains an "Euler path", if and only if it has exactly two vertices of odd degree.

Example: Determine the following graphs are Euler circuit or Euler path.

(1) G_1



(2) G_2



Sol:

(1) In the graph G_1, the vertices B and D are odd degree - 3 each. Hence an Euler path existed between "B & D."

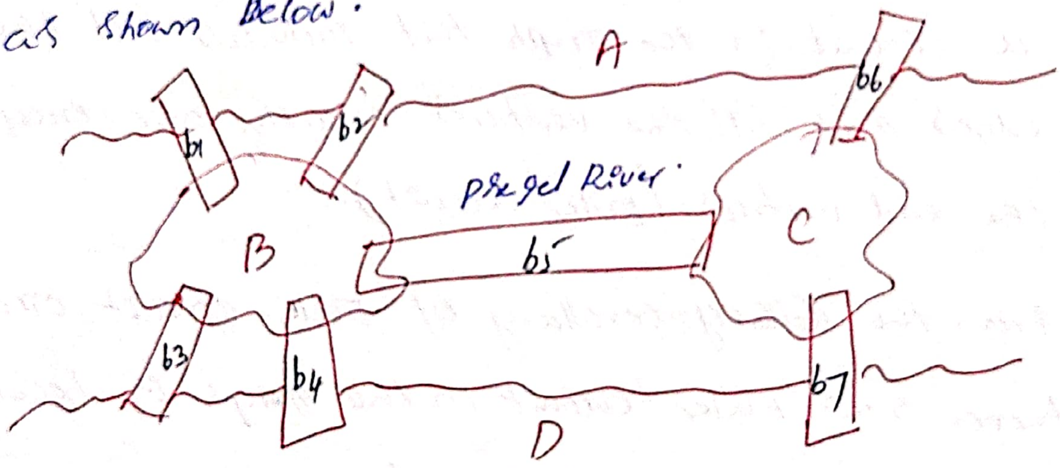
(3) In the graph - G_2, all the vertices are even degree. Hence, an Euler circuit existed.

Two practical problems for Eulerian and Hamiltonian circuits:

In order to see how graphs can be used to, denote or enterpreter or even solve some practical problems, we ~~first~~ first give below two well-known cases.

Seven bridge problem:

Two islands surrounded by a river are connected to each other and the river banks by seven bridges as shown below.



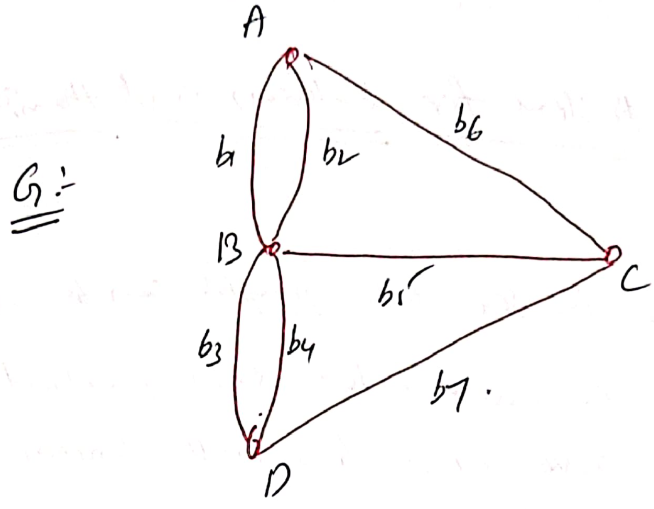
Can any one cross all the bridges shown in the above map exactly once and return to the starting point?

where $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ are bridges.

B, C — are islands

A, D - are two river banks respectetively.

The bridge configuration can be modeled as shown below.



The vertices represents the "locations". And the Edges represents the "bridges".

→ The Konigsberg bridge problem is reduced to finding a circuit in the graph that includes all the edges and all the vertices exactly once except the end vertices. (Euler circuit).

From the ~~Collary~~ Corollary of Euler ~~Circuit~~ circuit, there is no Euler circuit in the graph G below. there are four vertices of odd degree.

∴ There is no way to start at a given point cross each bridge exactly once and return to the starting point.

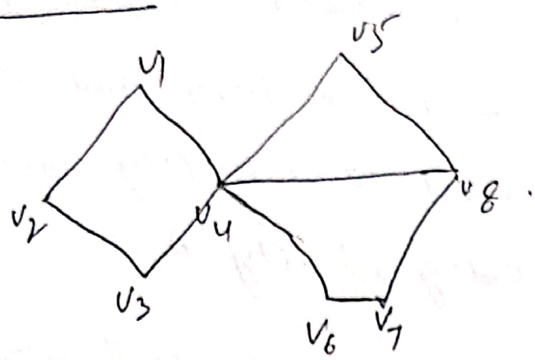
i.e Due to the above Euler's theorem, the seven bridge problem described earlier has no solution. i.e graph in the problem does not have "Eulerian circuit" because all vertices A, B, C, D have "Odd degrees".

5.12

Ex: Determine the following graphs are Euler circuit or Euler paths.

Sol:

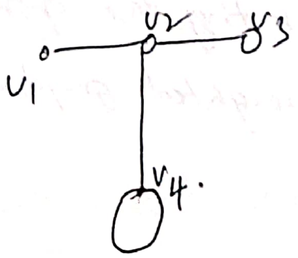
(a)



The graph has exactly "two" vertices of "odd degree". Thus graph is an Euler path but no Euler circuit.

V_4 — 5 and V_8 — 3.

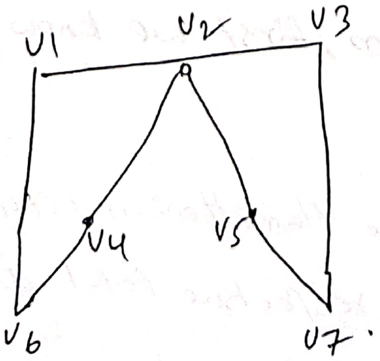
(b)



~~Each~~

Each vertices except V_4 has odd degree. Thus there is neither Euler ~~circcu~~ circuit nor a Euler path.

(c)



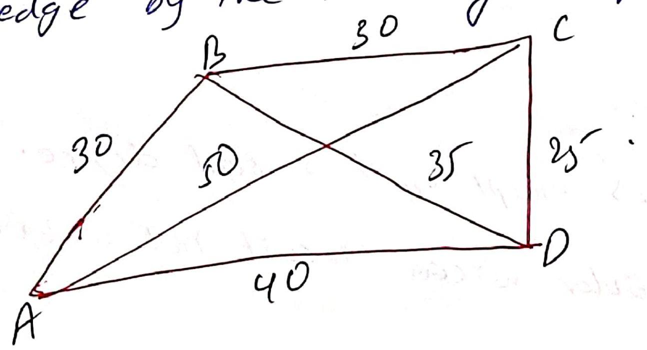
Each vertex has even degree. Thus Graph has an Euler circuit but not Euler Path.

Travelling Salesman problem:

Suppose the distance between each pair of vertices the cities A, B, C and D are given. And suppose a salesman must travel to each city exactly once, starting and ending at city A.

Which route from city to city will minimize the total travelling distance?

If we use vertices to denote cities, and put the distance between any two cities on the edge joining them, then we can representing the given knowledge by the following weighted graph.



Before considering this problem, first we know the Hamiltonian graphs.

Sol:- we need to find all the Hamiltonian circuits for the graph, calculate the respective total distance and then choose the shortest route.

route	total distance.
A → B → C → D → A	30 + 30 + 25 + 40 = 125
A → B → D → C → A	140
A → C → B → D → A	155
A → C → D → B → A	140
A → D → B → C → A	155
A → D → C → B → A	125.

- Hence the best route is either "ABCDA" or "ADCBA"

Hamiltonian graphs:

Hamiltonian path:-

- A path of graph 'G' is called a Hamiltonian path if it includes each vertex of G exactly once.

Hamiltonian circuit:

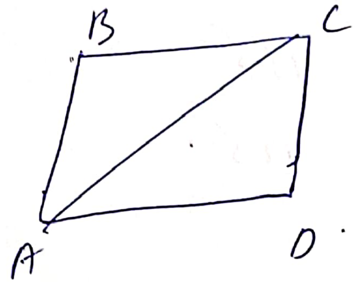
A circuit of graph 'G' is called a Hamiltonian circuit, if it includes each vertex of G' exactly once, except starting and end vertices. which appears twice.

Hamiltonian graph:-

A graph Containing Hamiltonian Circuit is called Hamiltonian graph.

Example:- Determine the following are Hamiltonian paths or circuits.

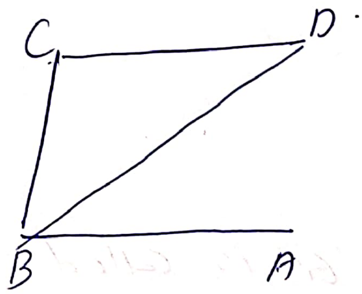
Sol:- G1



The graph G1, has Hamiltonian circuit namely

A - B - C - D - A.

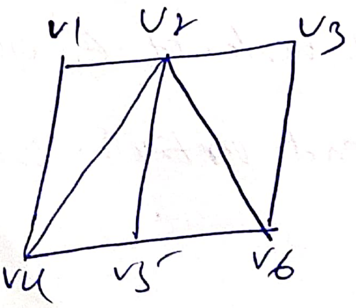
G2:



Graph G2 has a hamiltonian path namely A - B - C - D but not a hamiltonian circuit.

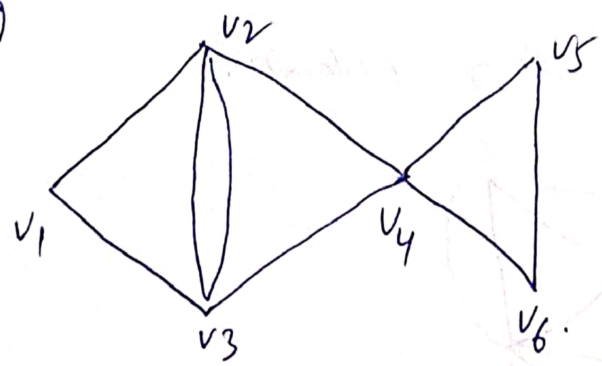
Ex:- Determine the following are Hamiltonian or Eulerian.

Sol:-



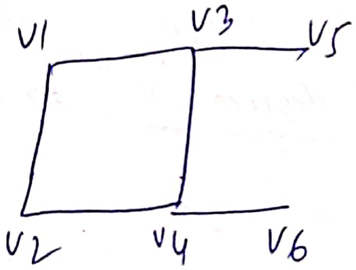
this is a Hamiltonian but not Eulerian.

(b)



Eulerian but not Hamiltonian.

(c)

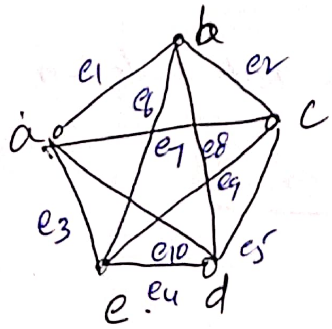


neither Eulerian nor Hamiltonian.

Examples:

① Show that the graph (G) is Eulerian?

G:-



sol:- ~~the~~ In the graph each vertex is even degree.

so, it exists a Euler circuit.

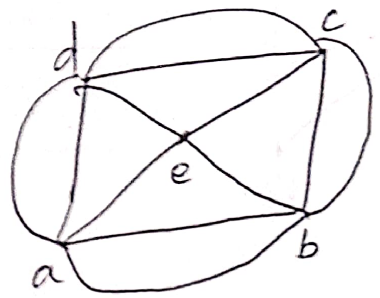
G is Eulerian with Euler circuit namely:

~~$a \xrightarrow{e_1} b \xrightarrow{e_8} d \xrightarrow{e_4} e \xrightarrow{e_3} a \xrightarrow{e_7} c \xrightarrow{e_9} e \xrightarrow{e_6} b - d - a$~~

$b \xrightarrow{e_1} a \xrightarrow{e_3} e \xrightarrow{e_4} d \xrightarrow{e_5} c \xrightarrow{e_2} b \xrightarrow{e_6} e \xrightarrow{e_9} c \xrightarrow{e_7} a \xrightarrow{e_{10}} d \xrightarrow{e_8} b.$

② Show that the graph is not Eulerian?

sol:



There are "four" vertices of degree "5" in the graph

deg (a) – 5

deg(b) – 5

deg(c) – 5

deg (d) – 5

~~deg~~

Therefore 'G' is not Eulerian.

Euler graphs for directed graphs

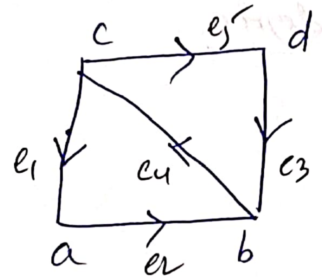
Corollary 1:

A directed multigraph 'G' has an Euler path if and only if it is unilaterally connected and the indegree of each vertex is equal to it out degree with the possible exception of two end vertices, for which it may be that the indegree of one is one larger than its outdegree and the indegree of the other is ~~[illegible]~~ one less than its out degree.

Corollary 2:-

A Directed multigraph 'G' has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its out degree.

Ex: Determine the following graph is Eulerian or not.



Sol:

V (G)	indegree	outdegree
a	1	1
b	2	1
c	1	2
d	1	1

The graph G is unilaterally connected and clearly the degrees of the vertices are satisfying corollary-1 on Euler path for directed multi graphs.

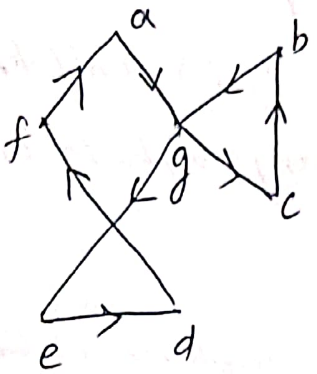
Therefore it has Euler path: $c \xrightarrow{e_1} a \xrightarrow{e_2} b \xrightarrow{e_4} c \xrightarrow{e_5} d \xrightarrow{e_3} b$.

As the degrees of the vertices are not satisfying the Corollary on Euler Circuit-2.

therefore the graph has no "Euler Circuit".

The given graph G has only Euler path.

Ex:- Find the Euler Path or circuit for the following directed graph.



Sol:-

V(G)	indegree	outdegree
a	1	1
b	1	1
c	1	1
d	1	1
e	1	1
f	1	1
g	2	2

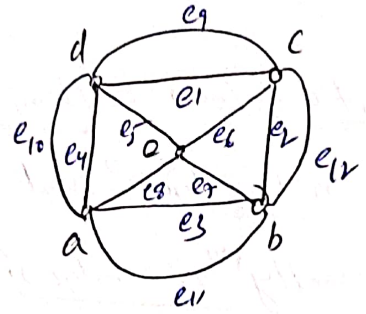
As the graph G is unilaterally connected and the degrees of all vertices are satisfying the corollary of Euler circuit for dimultigraphs.

Therefore given graph has Euler circuit.

Euler circuit:-

a — g — c — b — g — e — d — f — a.

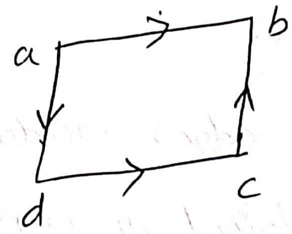
Ex:- Show that the graph 'G' is not Eulerian.



Sol: There are four vertices of degree 5 in the graph a, b, c, d.

Therefore 'G' is not Eulerian.

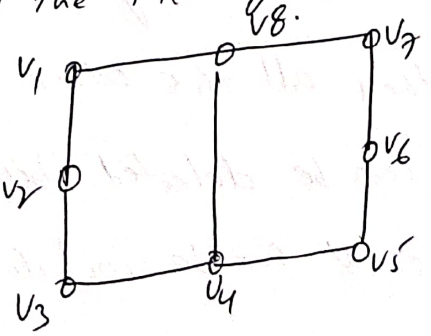
Ex:- Find Euler path or Euler circuit in the given graph.



Sol:- clearly 'd' is not reachable from 'b'. that is the given graph is not unilaterally connected.

Therefore the given graph has no Euler path and no Euler circuit.

Ex: Find the following graph has Hamiltonian circuit (HC).



Sol:- Hamiltonian cycle (HC) is

$V_1 - V_2 - V_3 - V_4 - V_5 - V_6 - V_7 - V_8 - V_1$.

Some Basic Rules for Constructing Hamiltonian path & cycles.

Rule1: If G has 'n' vertices, then a "Hamiltonian path" must contain "exactly n-1 edges" and a "Hamiltonian cycle" must contain exactly "'n' edges".

Rule2: If a vertex v in G has degree k, then Hamiltonian path must contain atleast one edge incident on 'v' and atmost two edges incident on v.

A HC (Hamiltonian circuit) contains exactly two edges incident on v.

In particular, both edges incident on a vertex of degree two will be contained in every HC.

Finally, there can not be three or more edges incident with one vertex in a HC.

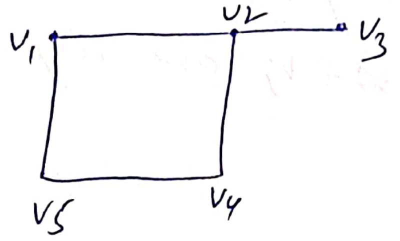
Rule3:- No cycles that does not contain all the vertices of G can be formed when building a HP or HC.

Rule4:- once the HC we are building has passed through a vertex V, then all the other unused edges incident on V can be deleted because only two edges incident on V can be included in a HC.

Example:

① Determine the following graph has Hamiltonian path or circuit.

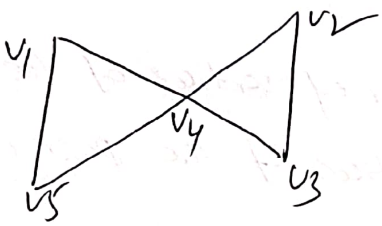
Sol:



HP: $v_1 - v_5 - v_4 - v_2 - v_3$.

HC: no HC since the graph has a vertex v_3 with degree 1.

Ex: Find HP and HC?



Sol: HP: $v_1 - v_5 - v_4 - v_2 - v_3$.

HC: it does not have a HC.

since $\deg(v_4) = 4$.

Strongly connected graph:

A Directed graph is said to be "strongly connected" if there is a path from v_i to v_j and from v_j to v_i. where v_i & v_j are any pair of vertices of the graph.

weakly connected;

A Directed graph is said to be "weakly connected" if there is a path between every two vertices in the underlying undirected graph. (Directions are disregarded) (or) removed.

unilaterally connected:

if for any pair of vertices of the graph, at least one of the vertices of the pair is reachable from the ~~other~~ other vertex.

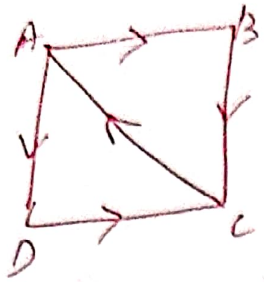
Note:

① unilaterally connected diagraph is weakly connected but weakly connected diagraph is not necessarly unilaterally connected.

② strongly connected diagraph is both ~~unilaterally~~ unilaterally and weakly connected.

Example:

G_1



find the graph is strongly connected or not.

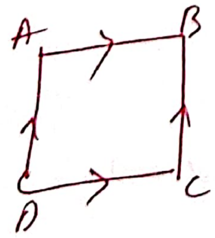
Sol: G_1 is strongly connected graph, as the possible pairs of vertices in G_1 are

(A, B), (A, C), (A, D), (B, C), (B, D), & (C, D).

And there is a path from first vertex to second vertex and second to the first in all the pairs.

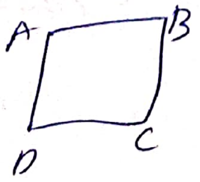
- path from (A - B) = A to B.
- " (B - A) = B - C - A.
- " (B - D) = B - C - A - D
- " (D - B) = D - C - A - B.
- " (A - D) = A to D
- " (A - C) = A - B - C. or A - D - C.
- " (C - D) = C - A - D.
- " (D - C) = D to C.

Ex:



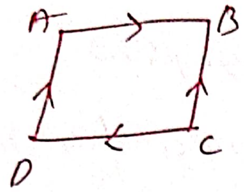
G_2

Sol: G_2 is weakly connected graph. because by the def. of weakly connected if we ignore the directections. the graph is.



There is no path from A - C. to C - A. and B - D, D - B.

Ex:



G_3

Sol: It is unilaterally connected since there is path from A to B, but there is no path from B to A.

similarly D − A but not A − D.
C − B " B − C.
D − B but not B − D.
C − A " A − C.

Multi Graph:-

Definition: It contains the multiple edges and loops is called the multi graph.

⇒ when graph 'G' is a multigraph, then a_{ij} denote the no: of edges (V_i, V_j), thus only adjacency matrix is used to represent multi graphs.

Adjacency Matrix:

for a multigraph 'G' consist of "n" vertices, an $n \times n$ adjacency matrix

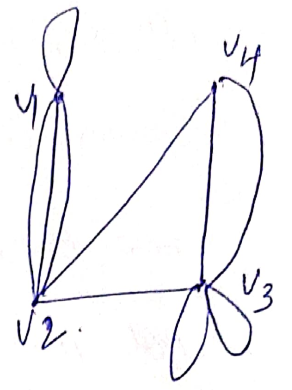
$A = [a_{ij}]$ is defined as,

$$a_{ij} = \begin{cases} m & \text{- if there are one or more edges between vertex } v_i \text{ \& } v_j \\ 0 & \text{otherwise} \end{cases}$$

m = no: of edges.

Ex: Specify the matrix representation of for a multigraph consist of 4 vertices.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$



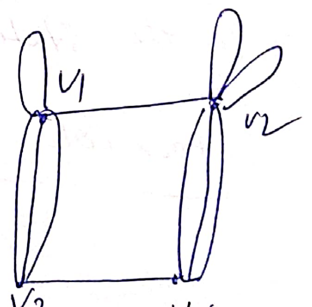
Ex: Determine the no: of loops and multiple edges in a multi graph from its adjacency matrix. Draw the graph.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix} \end{array}$$

Sol: 4 vertices $- v_1, v_2, v_3, v_4$.

loops $v_1 - 1$

$v_2 - 2$.



multiple loop are also multiple edges.

no: of edges $v_3 - v_1 = 3$

$v_4 - v_2 = 3$

$v_1 - v_1 = 1$

$v_2 - v_2 = 2$

totally "7" edges.

Four Color Theorem (without proof)

$\Rightarrow$ So far we discussed proper coloring of vertices and proper coloring of edges of graph.

$\Rightarrow$ now briefly consider the proper coloring of regions in planar graphs such that no two adjacent regions receive the same color.

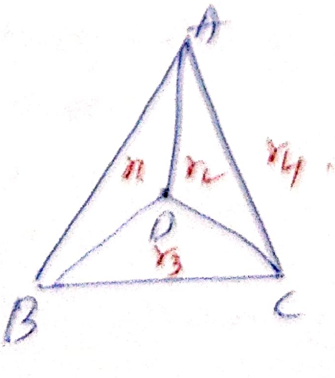
$\Rightarrow$ The four color problem states that every plane map however complex, can be colored with "four" colors in such a way that two adjacent regions get different colors.

$\Rightarrow$ This problem is solved by "Appel and Haken" in 1976. However this problem is infact equivalent to the statement of "conjecture".

Four color conjecture: every planar graph is 4-colorable.

Ex: The graph K_4 is planar graph and K_4 is 4-colorable. Show that.

Sol: K_4 — planar graph.



r_1 is adjacent to r_2, r_3 & r_4. So we can assign different colors.

$r_1 - c_1$

$r_2 - c_2$

$r_3 - c_3$

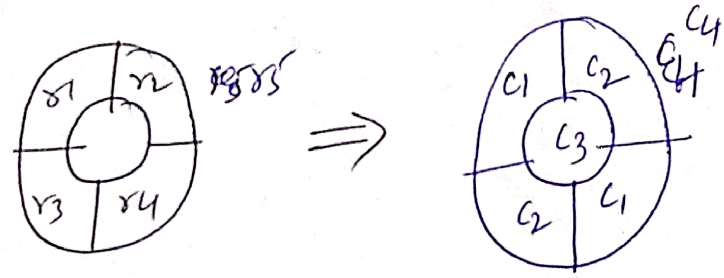
$r_4 - c_4$.

$\therefore K_4$ is 4-colorable.

Definition:

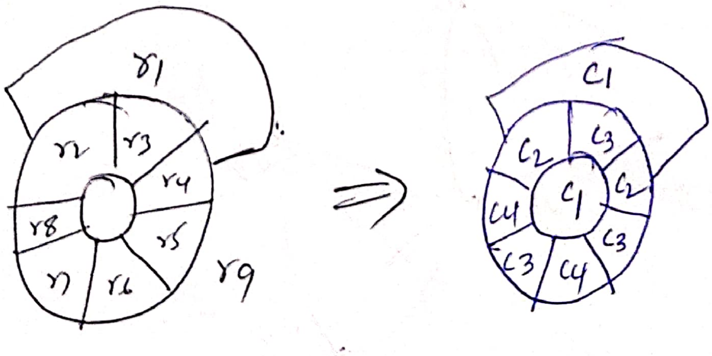
The four color theorem or four color map theorem, states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the map so that no two adjacent regions have the same color.

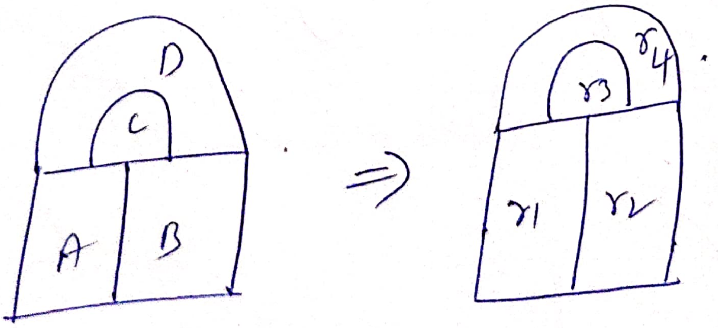
Example: Color the following planar graphs.

① 

$r_1 - c_1$
$r_2 - c_2$
$r_3 - c_2$
$r_4 - c_1$
$r_5 - c_4$.

4-colorable.

② 

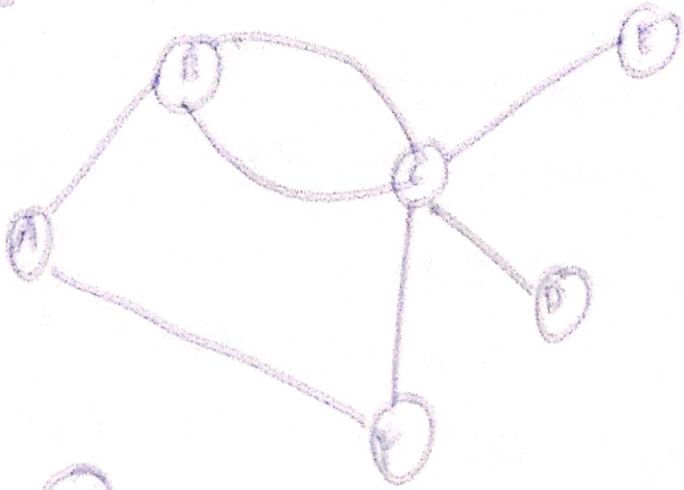
③ 

—— 0 ——

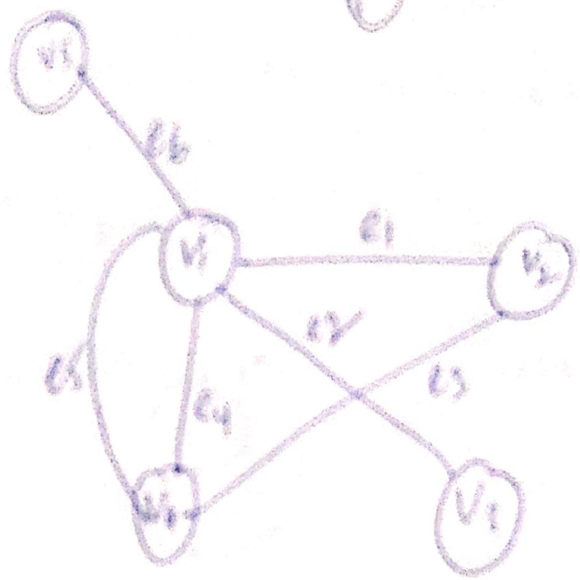
The End

Examples

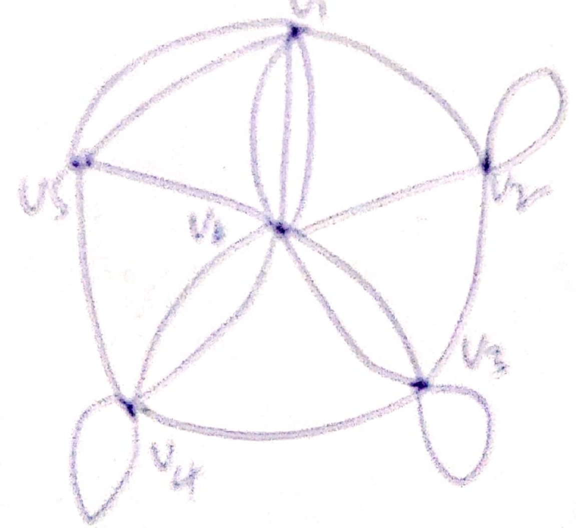
①



②



③



④