Recurrence Relation (UNIT - III)

Definition of Recurrence Relation

A Recurrence Relation is an equation that recursively defines a sequence or a rule that gives next term in the sequence as a function of the previous terms. when one or more initial terms are given.

Example:

The fibonacci sequence is defined by using the recurrence relation

$$\boxed{F_n = f_{n-1} + F_{n-2}}$$

with initial condition $\boxed{F_0 = 0, \; F_1 = 1}$

In the same way, we can find the next succeeding terms in the sequence such as.

$F_2 = F_1 + F_0$
$= 1 + 0$
$= 1$

$F_3 = F_2 + F_1$
$= 1 + 1$
$= 2$.

$F_4 = F_3 + F_2$
$= 2 + 1$
$= 3$

$F_5 = F_4 + F_3$
$= 3 + 2$
$= 5$

$F_6 = F_5 + F_4$
$= 5 + 3 = 8$

$F_7 = F_6 + F_5$
$= 8 + 5 = 13$

$F_8 = F_7 + F_6 = 13 + 8 = 21$

$\vdots$

$F_n = F_{n-1} + F_{n-2}, \; n \geq 2$.

we obtain the sequence of fibonacci numbers by using the above recurrence relation.

0, 1, 1, 2, 3, 5, 8, 13, 21, - - - .

Ex:- find out the sequence generating by the recurrence (b)
relation below.

sol:

(a) $T_n = 2 \cdot T_{n-1}$ with $T_1 = 4$.

sol:- Given Recurrence relation is.

$$\boxed{T_n = 2T_{n-1}}$$

Initial condition is: $T_1 = 4$.

$n=2$ $T_2 = 2T_{\cancel{2}-1}$

$= 2T_1 = 2 \times 4 = 8$.

$n=3$ $T_3 = 2T_{3-1}$

$= 2T_2 = 2 \times 8 = 16$.

$n=4$ $T_4 = 2T_{4-1}$

$= 2T_3 = 2 \times 16 = 32$

$n=5$ $T_5 = 2T_{5-1}$

$= 2T_4 = 2 \times 32 = 64$.

n:	2	3	4	5
T_n:	8	16	32	64.

Recursive sequence generated by the recurrence relation is: 4, 8, 16, 32, 64 ---

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(b) $T_n = 3 \cdot T_{n-1} - 4$ with $T_1 = 3$.

Sol:- Given recurrence relation is.

$$T_n = 3T_{n-1} - 4.$$

Initial condition is $T_1 = 3$.

$n=2 \Rightarrow T_2 = 3 \cdot T_{2-1} - 4$

$= 3 \cdot T_1 - 4$

$= 3 * 3 - 4 = 9 - 4 = 5.$

$n=3 \Rightarrow T_3 = 3 \cdot T_{3-1} - 4$

$= 3 \cdot T_2 - 4$

$= 3 * 5 - 4 = 15 - 4 = 11$

$n=4 \Rightarrow T_4 = 3 \cdot T_{4-1} - 4$

$= 3 T_3 - 4$

$= 3 * 11 - 4 = 33 - 4 = 29.$

$n=5 \Rightarrow T_5 = 3 \cdot T_{5-1} - 4$

$= 3 \cdot T_4 - 4.$

$= 3 * 29 - 4 = 87 - 4 = 83.$

Recursive sequence $= 3, 5, 11, 29, 83 \ldots$

(3) Find out the recurrence relation for the following sequence.

(a) 2, 6, 18, 14, 162 ....

(b) 20, 17, 14, 11, 18 ....

(c) 1, 3, 6, 10, 15, 21 ---

(a) 2, 6, 18, 54, 162 ----

Sol:- Given recursive sequence = 2, 6, 18, 54, 162 ...

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 2$ $T_4 = 54$

$T_2 = 6$ $T_5 = 162$

$T_3 = 18$

This is the initial conditions in the sequence.

Initial condition is : $T_1 = 2$.

$T_2 = 6 = 3 \cdot T_1 = 3 \cdot 2 = 6.$

$T_3 = 18 = 3 \cdot T_2 = 3 \cdot 6 = 18.$

$T_4 = 54 = 3 \cdot T_3 = 3 \cdot 18 = 54$

$T_5 = 162 = 3 \cdot T_4 = 3 \cdot 54 = 162.$

$T_2 = 6 = 3 \cdot T_1 = 3 \cdot 2 = 6$

$T_3 = 18 = 3 \cdot T_2 = 3.$

$\boxed{T_n = 3 \cdot T_{n-1}}$

$\vdots$

$T_n = 3 \cdot T_{n-1}.$ with $T_1 = 2$.

$\therefore$ Recurrence Relation for the sequence, is

2, 6, 18, 54, 162 ---- is $\boxed{T_n = 3T_{n-1}}$ with $T_1 = 2$.

$\downarrow$ Prove that.

(b) 20, 17, 14, 11, 8 ----

Sol:- Given recurrence ~~relation~~ sequence is

= 20, 17, 14, 11, 8 ---.

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 20$ $T_3 = 14$ $T_5 = 8$.

$T_2 = 17$ $T_4 = 11$

Initial conditions $T_1 = 20$.

$$T_2 = 17 = T_1 - 3 = 20 - 3 = 17.$$

$$T_3 = 14 = T_2 - 3 = 17 - 3 = 14.$$

$$T_4 = 11 = T_3 - 3 = 14 - 3 = 11.$$

$$T_5 = 8 = T_4 - 3 = 11 - 3 = 8.$$

$\vdots$

$\boxed{T_n = T_{n-1} - 3}$ with initial condition $T_1 = 20$.

(c) 1, 3, 6, 10, 15, 21 ....

Sol: Given reference ~~relation~~ sequence is 1, 3, 6, 10, 15, 21.

Given recurrence is the recursive sequence.

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, $T_6 = 21$.

Initial conditions $T_1 = 1$.

$$T_2 = 3 = T_1 + 2 = 1 + 2 = 3.$$

$$T_3 = 6 = T_2 + 3 = 3 + 3 = 6.$$

$$T_4 = 10 = T_3 + 4 = 6 + 4 = 10.$$

$$T_5 = 15 = T_4 + 5 = 10 + 5 = 15$$

$$T_6 = 21 = T_5 + 6 = 15 + 6 = 21.$$

$\vdots$

$\boxed{T_n = T_{n-1} + n.}$ With initial condition $T_1 = 1$.

Note:- The next term depends on previous one term called first order relation.

Example: Find the first five terms of the sequence defined by each of the following recurrence relations and initial conditions.

(a) $a_n = a_{n-1}^2$, $a_1 = 2$.

The given recurrence relation is $a_n = a_{n-1}^2$.

Given that $a_1 = 2$.

$a_2 = a_{2-1}^2$

$= a_1^2 = 2^2 = 4$

$a_3 = a_{3-1}^2$

$= a_2^2 = 4^2 = 16.$

$a_4 = a_{4-1}^2$

$= a_3^2 = 16^2 = 256.$

$a_5 = a_{5-1}^2$

$= a_4^2 = (256)^2 = 65,356.$

Hence the first five terms of sequence are:

2, 4, 16, 256 and 65,356.

(b) $a_n = n a_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$.

Sol:- Given recurrence relation

~~$a_n = 0$~~

$$\boxed{a_n = n a_{n-1} + n^2 a_{n-2}}$$

Initial Conditions. $a_0 = 1,\ a_1 = 1.$

$$a_2 = 2a_{2-1} + 2^2 a_{2-2}$$

$$= 2a_1 + 4a_0.$$

$$= 2\times1 + 4\times1 = 2+4 = 6.$$

$$a_3 = 3a_{3-1} + 3^2 a_{3-2}.$$

$$= 3a_2 + 9a_1$$

$$= 3\times6 + 9\times1 = 18+9 = 27.$$

$$a_4 = 4a_{4-1} + 4^2 a_{4-2}$$

$$= 4a_3 + 16a_2$$

$$= 4\times27 + 16\times6 = 204.$$

Thus the first five terms of the sequence are.

1, 1, 6, 27 and 204.

(2) $a_n = a_{n-1} + a_{n-3}$ $a_0 = 1,\ a_1 = 2,\ a_2 = 0.$

Sol: Given recurrence relation.

$$\boxed{a_n = a_{n-1} + a_{n-3}}$$

Initial Conditions $a_0 = 1,\ a_1 = 2,\ a_2 = 0.$

$$a_3 = a_{3-1} + a_{3-3}$$

$$= a_2 + a_0 = 0+1 = 1.$$

$$a_4 = a_{4-1} + a_{4-3} = a_3 + a_1 = 1+2 = 3.$$

$$a_5 = a_{5-1} + a_{5-3} = a_4 + a_2 = 3+0 = 3.$$

Thus, the first five terms of the sequence are

~~0,2~~ 1, 2, 0, 1, 2, 4 ....

Ex: let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3a_{n-2}$ for $n = 2, 3, \ldots$ and let $a_0 = 1$ and $a_1 = 2$. What are the values of a_2 and a_3.

Sol:- The given recurrence relation is

$$\boxed{a_n = a_{n-1} + 3a_{n-2}}, \quad n = 2, 3 \ldots$$

$$a_2 = a_{2-1} + 3a_{2-2}.$$

$$= a_1 + 3a_0 = 2 + 3 \times 1 = 2+3 = 5.$$

$$\therefore a_2 = 5.$$

$$a_3 = a_{3-1} + 3a_{3-2}$$

$$= a_2 + 3a_1 = 5 + 3 \cdot 2 = 5 + 6 = 11.$$

$$\therefore a_3 = 11.$$

Example: Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 2n - 9 \text{ if.}$$

(i) $a_n = -n+2$ (ii) $a_n = 3(-1)^n + 2^n - n + 2$.

Sol: The given recurrence relation is

$$\boxed{a_n = a_{n-1} + 2a_{n-2} + 2n - 9.}$$

(i) $a_n = -n+2$.

$$a_{n-1} = -(n-1) + 2$$

$$a_{n-2} = -(n-2) + 2.$$

... recurrence relation

$= a_{n-1} + 2a_{n-2} + 2n - 9.$

$= [-(n-1)+2] + 2[-(n-2)+2] + 2n - 9.$

$= -n+1+2+2[-n+2+2] + 2n - 9$

$= -n+1+2 - \cancel{2n} + 4 + 4 + \cancel{2n} - 9.$

$= -n+3+8-9$

$= -n + 11 - 9$

$= -n + 2.$

$= a_n. = L.H.S.$

$\therefore a_n = -n+2$ is a solution of the recurrence relation.

(ii) $a_n = 3(-1)^n + 2^n - n + 2.$

$R.H.S = a_{n-1} + 2a_{n-2} + 2n - 9.$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2.$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2.$

$= 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2] + 2n - 9.$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2(2^{n-2}) + 2^{n-1} - n + 1 + 2 - 2n + 4 + 4 + 2n - 9.$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2(2^{n-2}) + 2^{n-1} - n + 2.$

$$= 6\cdot\frac{(-1)^n}{(-1)^2} + 3\cdot\frac{(-1)^n}{(-1)^1} + 2\cdot\frac{2^n}{2^2} + \frac{2^n}{2^1} - n + 2$$

$$= 6(-1)^n - 3(-1)^n + \frac{2^n}{2} + \frac{2^n}{2} - n + 2$$

$$\cancel{= 6(-1)^n - 3(-1)^n + 2\cdot\frac{2^n}{2} - n + 2}$$

$$\cancel{= 6(-1)^n - 3(-1)^n - n + 2}.$$

$$2$$

$$= 6(-1)^n - 3(-1)^n + \cancel{2}\cdot\frac{2^n}{\cancel{2}} - n + 2$$

$$= 6(-1)^n - 3(-1)^n + 2^n - n + 2.$$

$$= 3(-1)^n + 2^n - n + 2.$$

$$= a_n. = \text{L.H.S.}$$

$\therefore a_n = 3(-1)^n + 2^n - n + 2$ is a solution of the given recurrence relation.

Example:

$\Rightarrow$ let $a_n = 2^n + 5(3^n)$, for $n = 0, 1, 2 \ldots$

(i) Find a_0, a_1, a_2, a_3 and a_4.

(ii) show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$ and $a_4 = 5a_3 - 6a_2$.

(iii) ST $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

Solve the recurrence relations 3.6

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2, a_0 = 1, a_1 = 0.$$

Sol: The given recurrence relation is

$$a_n - 5a_{n-1} + 6a_{n-2} = 0.$$

The characteristics equation of the recurrence relation

is $r^2 - 5r + 6 = 0$.

Characteristic roots: $r(r-2) + 3(r+1)$.

$$r(r-3) - 2(r-3)$$

$$r^2 - 3r - 2r + 6.$$

$\Rightarrow r(r-3) - 2(r-3)$

$$r-2 = 0 \Rightarrow r = 2$$

$$r-3 = 0 \Rightarrow r = 3.$$

Hence, the solution is

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n.$$

Initial conditions are $a_0 = 1, a_1 = 0$.

now $a_0 = c_1 \cdot 2^0 + c_2 \cdot 3^0$.

$$\Rightarrow c_1 + c_2 = 1. \quad \text{——— (1)}$$

$$a_1 = c_1 \cdot 2^1 + c_2 \cdot 3^1$$

$$2c_1 + 3c_2 = 0. \quad \text{——— (2)}$$

Solving (1) & (2): $c_1 = 1 - c_2$.

$$2(1-c_2) + 3c_2 = 0.$$

$$2 - 2c_2 + 3c_2 = 0$$

$$2 + c_2 = 0 \Rightarrow c_2 = -2.$$

$$c_1 = 1 + 2 = 3$$

Hence the sol. is

$$a_n = 3 \cdot (2^n) - 2 \cdot (3^n).$$

for $n \geq 2$.

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Example Solve the recurrence relation is

$$a_n - 8a_{n-1} + 16a_{n-2} = 0 \text{ for } n \geq 2, a_0 = 16, a_1 = 80$$

Sol:- The given recurrence relation is.

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 8r + 16 = 0.$$

Characteristic roots.

$$r(r-4) \cancel{+} -4(r-4).$$

$$\Rightarrow (r-4)^2 = 0.$$

$$\Rightarrow r = 4, 4. \quad \text{roots are equal.}$$

$$a_n = (c_1 + c_2 n) r^n.$$

Hence the solution is.

$$\boxed{a_n = \cancel{4} c_1 4^n + c_2 \times 4^n \cdot n.}$$ where c_1 & c_2 are arbitrary constants.

Initial conditions:

$$a_0 = 16 \Rightarrow c_1 \cdot 4^0 + c_2 \cdot 4^0 \times 0$$

$$\Rightarrow c_1 + 0 = 16. \Rightarrow c_1 = 16.$$

$$a_1 = 80 \Rightarrow c_1 \cdot 4^1 + c_2 \cdot 4^1 \times 1.$$

$$\Rightarrow 4c_1 + 4c_2 = 80.$$

$$\Rightarrow 4 \times 16 + 4c_2 = 80$$

$$4c_2 = 80 - 64$$

$$4c_2 = 16 \Rightarrow c_2 = 16/4 = 4.$$

Hence the unique solution is

$$a_n = 16(4^n) + 4n(4^n) = 4^{n+2} - n4^{n+1}$$

$$a_n = 4^1 \times 4^{n+1} - n4^{n+1}$$

$$= 4^{n+1}(4-n), \; n \geq 2.$$

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UNIT III : Reccurence Relations

Defⁿ: A reccurence relation is an equation that recursively defines a sequence on a rule that gives next term in the sequence as a funcⁿ of previous terms when one/more initial terms are given.

Recurrence Relations are two Types:

1) Linear Recurrence Relation/ Homogeneous RR

2) Non-Linear/Non- Homogeneous RR

To solve a Recurrence Relation, we have 3 approaches:

1) Iteration Method

2) Characteristic root Method

3) Generating Method

1) Find out the sequence generating by the recurrence relation given below

$$T_n = 2T_{n-1} \quad \text{with } T_1 = 4$$

Sol: $T_1 = 4$

$n = 1 \Rightarrow T_1 = 4$

$n = 2 \Rightarrow T_2 = 2 * T_1 = 2 \times 4$
$= 8$

$n = 3, \quad T_3 = 2 \cdot T_2 = 2 \times 8 = 16$

$n = 4, \quad T_4 = 2 \cdot T_{4-1} = 2 \times T_3 = 32$

$n = 5, \quad T_5 = 2 \cdot T_4 = 2 \times 32 = 64$

$n = 6, \quad T_6 = 2 \cdot T_5 = 2 \times 64 = 128$

$n = 7, \quad T_7 = 2 \cdot T_6 = 2 \times 128 = 256$

Sequence of ~~pro~~ the generated terms: 4, 8, 16, 32, 64, 128 ....

Q) $T_n = T_{n-1} - 4$ with $T_1 = 3$

Sol: $n = 1, \quad T_1 = 3$

$n = 2, \quad T_2 = 3 \cdot T_{2-1} - 4 = 9 - 4$

$= 5$

$n = 3, \quad T_3 = 3 \cdot T_{3-1} - 4 = 3 \times 5 - 4$

$T_3 = 11$

$n = 4 \Rightarrow T_4 = 3 \cdot T_{4-1} - 4 = 3 \times 11 - 4$

$= 29$

$n = 5 \Rightarrow T_5 = 3 \times T_{5-1} - 4 = 3 \times 29 - 4$

$= 83$

$n = 6 \Rightarrow T_6 = 3 \times T_{6-1} - 4 = 3 \times 83 - 4$

$= 245$

Sequence: 3, 5, 11, 29, 83, 245, ...

Q) Find the reccurence relation for the following sequence

i) 2, 6, 18, 54, 162 ...

Sol: Given sequence; 2, 6, 18, 54, 162

$T_1 = 2 \longrightarrow$ Initial Condition

$T_2 = 6 \Rightarrow$

$T_3 = 18$

$T_4 = 54$

$T_5 = 162$

$\vdots$

$T_n = 3 * T_{n-1}$

ii) 20, 17, 14, 11, 8. ...

$T_1 = 20$ Initial Condition

$T_2 = 17 \Rightarrow T_{2-1} - 3 = T_1 - 3$

$T_3 = 14 \Rightarrow T_{3-1} - 3 = T_2 - 3$

$T_4 = 11 \Rightarrow T_{4-1} - 3 = T_3 - 3$

$\vdots$

$T_n = T_{n-1} - 3$

iii) 1, 3, 6, 10, 15, 21 . . . .

$T_1 = 1$ Initial Condition

$T_2 = 3 \Rightarrow T_1 + 2$

$T_3 = 6 \Rightarrow T_2 + 3$

$T_4 = 10 \Rightarrow T_3 + 4$

$T_5 = 15 \Rightarrow T_4 + 5$

$\vdots$

$T_n = T_{n-1} + n$

Q) Find the first 5 terms of sequence defined by each of the following RR & initial condn.

i) $a_n = a_{n-1}^2$, $a_1 = 2$

$a_1 = 2$

$a_2 = a_1^2 = 2 \times 2 = 4$

$a_3 = a_2^2 = 4 \times 4 = 16$

$a_4 = a_3^2 = 16 \times 16 = 256$

$a_5 = a_4^2 = 256 \times 256 = 65536$

ii) $a_n = n a_{n-1} + n^2 a_{n-2}$ with $a_0 = 1$, $a_1 = 1$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2 \cdot a_1 + 2^2 \cdot a_0 = 2 \times 1 + 4 \times 1$

$= 6$

$a_3 = 3 \cdot a_2 + 3^2 \cdot a_1 = 3 \times 6 + 9 \times 1$

$= 27$

$a_4 = 4 \cdot a_3 + 4^2 \cdot a_2 = 4 \times 27 + 16 \times 6$

$= 204$

$a_5 = 5 \cdot a_4 + 5^2 \cdot a_3 = 5 \times 204 + 25 \times 27$

$= 1695$

iii) $a_n = a_{n-1} + a_{n-3}$ with $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

$a_0 = 1$

$a_1 = 2$

$a_2 = 0$

$a_3 = a_2 + a_0 = 0 + 1 = 1$

$a_4 = a_3 + a_1 = 1 + 2 = 3$

$a_5 = a_4 + a_2 = 3 + 0 = 3$

Q) Determine whether the sequence a_n is a solution of recurrence relation $a_n = a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

i) if $a_n = -n + 2$

ii) if $a_n = 3(-1)^n + 2^n - n + 2$

Sol:

i) $a_n = -n+2$

$a_{n-1} = -(n-1)+2 = -n+3$

$a_{n-2} = -(n-2)+2 = -n+4$

R.H.S $\Rightarrow a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

$= (-n+3) + 2(-n+4) + 2n - 9$

$= -n+3 + -2n+8 + 2n - 9$

$= -n+2$

$= a_n$

$\therefore a_n = a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

ii) $a_n = 3(-1)^n + 2^n - n + 2$

$a_{n-1} = 3(-1)^{(n-1)} + 2^{(n-1)} - (n-1) + 2$

$-3(-1)^n + 2^{n-1} - n + 3$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$= 3(-1)^n + 2^{n-2} - n + 4$

R.H.S $\Rightarrow$ $a_{n-1} + 2\cdot a_{n-2} + 2n - 9$

$$= -3(-1)^n + 2^{n-1} - n + 3 + 2\left(3(-1)^n + 2^{n-2} - n + 4\right)$$

$$= -3(-1)^n + 2^{n-1} - n + 3 + 6(-1)^n + 2^{n-1} - 2n + 8$$

$$= 3(-1)^n + 2^n - 3n + 2n + 11 - 9$$

$$= 3(-1)^n + 2^n - n + 2$$

$$= a_n$$

Q) Let $a_n = 2^n + 5(3^n)$ for $n = 0, 1, 2 \ldots$

i) find a_0, a_1, a_2, a_3 and a_4

ii) S.T $a_2 = 5a_1 - 6a_0$

$a_3 = 5a_2 - 6a_1$

iii) S.T $a_n = 5a_{n-1} - 6a_{n-2}$ $\forall$ integers n with $n \geq 2$

Sol: Given, $a_n = 2^n + 5(3^n)$

$n = 0 \Rightarrow a_0 = 2^0 + 5(3^0) = 1 + 5 = 6$

$n = 1 \Rightarrow a_1 = 2^1 + 5(3^1) = 2 + 5(3)$

$\boxed{a_1 = 17}$

$n = 2 \Rightarrow a_2 = 2^2 + 5(3^2) = 4 + 5(9)$

$\boxed{a_2 = 49}$

$n = 3 \Rightarrow a_3 = 2^3 + 5(3^3) = 8 + 5(27)$

$$\boxed{a_3 = 143}$$

$n = 4 \Rightarrow a_4 = 2^4 + 5(3^4) = 16 + 5(81)$

$$\boxed{a_4 = 421}$$

i)

$a_0 = 6$

$a_1 = 17$

$a_2 = 49$

$a_3 = 143$

$a_4 = 421$

ii) R.H.S $\Rightarrow 5a_1 - 6a_0$

$= 5(17) - 6(6)$

$= 85 - 36$

$= 49$

$= a_2$

$\therefore a_2 = 5a_1 - 6a_0$

R.H.S $\Rightarrow 5a_2 - 6a_1$

$= 5(49) - 6(17)$

$= 245 - 102$

$= 143$

$= a_3$

$\therefore a_3 = 5a_2 - 6a_1$

iii) $a_n = 5a_{n-1} - 6a_{n-2}$

$a_n = 2^n + 5(3^n)$

$a_{n-2} = 2^{(n-2)} + 5(3^{n-2})$

$a_{n-1} = 2^{n-1} + 5(3^{n-1})$

R.H.S $\Rightarrow$ $5a_{n-1} - 6a_{n-2}$

$= 5\left[2^{n-1} + 5(3^{n-1})\right] - 6\left[2^{n-2} + 5(3^{n-2})\right]$

$= 5.2^{n-1} + 25(3^{n-1}) - 6.2^{n-2} - 30.3^{n-2}$

$= 5.2^{n-1} + 25(3^{n-1}) - \cancel{6}^{3}.\frac{2^{n-1}}{\cancel{2}} - \cancel{30}^{10}.\frac{3^{n-1}}{\cancel{3}}$

$= 2.2^{n-1} + 10.3^{n-1}$

$= 2^n + 15.3^{n-1}$

$= 2^n + 5(3^n)$

$= a_n$

$\therefore a_n = 5a_{n-1} - 6a_{n-2}$

Solving the Reccurrence Relation using Characteristic Root

Order of a Recurrence Relation: Difference between highest subscript and lowest subscript

Ex: $a_n + a_{n-1} = 0 \longrightarrow n-(n-1) = n-n+1 = 1$ Order

$a_n + a_{n-1} + a_{n-2} = 0$

$a_n + 2a_{n-1} = r^2$

Method of Characteristic Roots

Steps:

1) Write the characteristic Equation for the given rearrence relation

2) Find the roots and lets roots be $r_1, r_2, r_3 \ldots r_n$

3) If r_1 and r_2 are Real & Distinct then the solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 & α_2 are arbitrary constants.

4) If r_1 and r_2 are Real and Equal then the sol^n is

$$a_n = (\alpha_1 + \alpha_2 n) r^n$$

5) If r_1 and r_2 are Complex then the sol^n is

$$a_n = r^n (\alpha_1 \cos n\phi + \alpha_2 \cos n\phi)$$

Q) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ $a_0 = 1$, $a_1 = 0$

Sol: The given recurrence relation is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \longrightarrow ①$$

Order = 2

Characteristic Eqn. $r^2 - 5r + 6 = 0 \longrightarrow ②$

$$(r-2)(r-3) = 0$$

$$r = 2, 3$$

$$r_1 = 2 \qquad r_2 = 3$$

The roots are real and distinct

Then the soln is: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\therefore a_n = \alpha_1 2^n + \alpha_2 3^n \longrightarrow ③$$

Initial conditions are $a_0 = 1$, $a_1 = 0$

$n = 0 \Rightarrow$ $a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0$

$$1 = \alpha_1 + \alpha_2 \longrightarrow (i)$$

$n = 1 \Rightarrow$ $\alpha_1 \cdot 2^1 + \alpha_2 \cdot 3^1 = a_1$

$$2\alpha_1 + 3\alpha_2 = 0 \longrightarrow (ii)$$

Solving ① & (ii)

$\alpha_2 = 1 - \alpha_1$ $\Rightarrow$ $2\alpha_1 + 3(1-\alpha_1) = 0$

$$2\alpha_1 + 3 - 3\alpha_1 = 0$$

$$\boxed{\alpha_1 = 3}$$

$\alpha_2 = 1 - 3$

$$\boxed{\alpha_2 = -2}$$

Substitute α_1, α_2 in (3)

$\therefore$ The sol^n is $\boxed{a_n = 3(2^n) - 2(3^n)}$

$n=0 \Rightarrow a_0 = 3(2^0) - 2(3^0) = 1$

$n=1 \Rightarrow a_1 = 3(2^1) - 2(3^1) = 0$

2) Solve the following recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ for $n \geq 2$ with $a_0 = 16$, $a_1 = 80$

Sol: Given recurrence relation:

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

Order = 2

Ch. Eq: $r^2 - 8r + 16 = 0$

$$(r-4)(r-4) = 0$$

$$r = 4, 4$$

The roots are real and equal.

$$a_n = (\alpha_1 + \alpha_2 n) 4^n$$

$n=0 \Rightarrow (\alpha_1 + \alpha_2 \cdot 0) 4^0 = a_0$

$\alpha_1 = 16$

$n=1 \Rightarrow (\alpha_1 + \alpha_2) 4^1 = a_1$

$(16 + \alpha_2) \cancel{4} = \overset{20}{\cancel{80}}$

$\alpha_2 = 4$

$\therefore a_n = (16 + 4n) 4^n$

3.) Solve the following: $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$

for $n \geq 3$ with $a_0 = 3$, $a_1 = 6$ & $a_2 = 0$

Sol:

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

Order = 3

ch. Eqn: $r^3 - 2r^2 - r + 2 = 0$

$$r = -1, 1, 2$$

$$a_n = \alpha_1(-1)^n + \alpha_2(1)^n + \alpha_3(2)^n$$

$n = 0 \Rightarrow$ $\alpha_1(-1)^0 + \alpha_2(1)^0 + \alpha_3(2)^0 = a_0$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

$n = 1 \Rightarrow$ $\alpha_1(-1)^1 + \alpha_2(1)^1 + \alpha_3(2)^1 = a_1$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6$$

$n = 2 \Rightarrow$ $\alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 = a_2$

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$$\alpha_1 = -3, \quad \alpha_2 = 3, \quad \alpha_3 = 0$$

$a_0 = 3$

$a_1 = 6$

$a_2 = 0$

$$a_n = -3(-1)^n + 3(1)^n + 0(2)^n$$

$$= -3(-1)^n + 3(1)^n$$

4) $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$ $\quad a_0 = -1,\ a_1 = 8$

5) $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ $\quad a_0 = 5,\ a_1 = 12$

4 Ans:) Given,

$$a_n + a_{n-1} - 6a_{n-2} = 0$$

Order = 2

Ch. Eqn: $r^2 - r - 6 = 0 \longrightarrow (1)$

$$r = \frac{+1 \pm \sqrt{1-4(-6)}}{2(1)} = \frac{1 \pm 5}{2}$$

$$r = 3, -2$$

$$a_n = \alpha_1 3^n - \alpha_2 2^n \longrightarrow (2)$$

$n = 0$ in (1) $\Rightarrow \alpha_1 3^0 - \alpha_2 2^0 = a_0$

$$\alpha_1 - \alpha_2 = -1$$

$$\alpha_1 = -1 + \alpha_2$$

$n = 1$ in (1) $\Rightarrow \alpha_1 \cdot 3^1 - \alpha_2 2^1 = a_1$

$$3\alpha_1 - 2\alpha_2 = 12$$

$$3(-1 + \alpha_2) - 2\alpha_2 = 12 \qquad \boxed{\alpha_1 = 14}$$

$$-3 + 3\alpha_2 - 2\alpha_2 = 12$$

$$\boxed{\alpha_2 = 15}$$

$$a_n = 14(3^n) - 15(2^n)$$

5 Ans: Given,

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Order = 2

Ch. Eqn: $r^2 - 6r + 9 = 0 \longrightarrow ①$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

$$a_n = (\alpha_1 + \alpha_2 n) 3^n \longrightarrow ②$$

$n = 0$ in ② $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 0) 3^0 = a_0$

$$\boxed{\alpha_1 = 5}$$

$n = 1$ in ② $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 1) 3^1 = a_1$

$$(5 + \alpha_2) 3 = 12$$

$$15 + 3\alpha_2 = 12$$

$$3\alpha_2 = 12 - 15$$

$$\boxed{\alpha_2 = -1}$$

α_1, α_2 in ② | $a_n = (5 - n) 3^n$

$a_0 = 5$ ✓
$a_1 = 12$ ✓

Solution of Non Homogeneous Recurrence Relation

* If recurrence relation is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k} + G(n) \longrightarrow (1)$$

The general solution of this non-homogeneous recurrence relation

$$a_n = a_n^{(h)} + a_n^{(p)}$$

h – homogeneous solⁿ

p – particular solⁿ

Steps:

(i) We obtain the homogeneous solution

first we write associated homogeneous recurrence relation

$$f(n) = 0$$

$$(1) \Rightarrow \quad a_n + c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k a_{n-k} = 0$$

(ii) We obtain the particular solution

There is no general procedure for obtaining the particular sol. of a recurrence relation.

However, if $f(n)$ has any one of the following forms:

1.) Polynomial in 'n' $(2+3n)$ / $2n+(3n-1)$

2.) Constant $(2, 3, 4 \dots$ etc$)$

3.) Powers of constant $(2^n, 3^n \dots)$

Particular Solution for f(n)

S.No.	f(n)	Form of Particular Solution
1.)	Constant, C	Constant, d
2.)	Linear func^n, $c_0 + c_1 n$	$d_0 + d_1 k$
3.)	N	$d_0 + d_1 k$
4.)	n^2	$d_0 + d_1 k + d_2 k^2$
5.)	m^{th} degree polynomial $c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$	$d_0 + d_1 k + d_2 k^2 + \cdots + d_m k^m$
6.)	$r^n \quad r \in R$	$d r^n$

(iii) The general solution of the recurrence relation can be written as sum of homogeneous solution and all the particular solutions.

Q) Solve the following recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \text{ for } n \geq 2 \quad a_0 = 1$$
$$a_1 = 2$$

Sol: The given recurrence relation is:

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \longrightarrow (1)$$

It is a Non linear RR of 2nd order

The general solution of the reccurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

(i) To obtain $a_n^{(h)}$, put $f(n) = 0$

(1) $\Rightarrow$ $a_n + 4a_{n-1} + 4a_{n-2} = 0$

Characteristic Eqn: $r^2 + 4r + 4 = 0$

$$r = \frac{-4 \pm \sqrt{16 - 4(4)}}{2(1)} = \frac{-4}{2} = -2, -2$$

Roots are real & equal

general solⁿ: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)(-2)^n \longrightarrow (2)$

Put $n = 0$ in (2) $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 0)(-2)^0 = a_0$

$$\alpha_1 = 1$$

$n = 1$ $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 1)(-2)^1 = a_1$

$$(1 + \alpha_2)(-2) = 2$$
$$\alpha_2 = -2$$

$$a_n^{(h)} = (1-2n)(-2)^n$$

(ii) To obtain $a_n^{(p)} = d \longrightarrow$ (3) ($f(n) = A$ constant)

Put (3) in (1)

$$(1) \Rightarrow d + 4d + 4d = 8$$

$$9d = 8$$

$$\boxed{d = \frac{8}{9}}$$

(iii) The general solution is $a_n^{(h)} + a_n^{(p)}$

$$a_n = (\alpha_1 + \alpha_2 n)(-2)^n + \frac{8}{9} \longrightarrow (4)$$

Put $n=0$ in (4) $\Rightarrow (\alpha_1 + 0)(-2)^0 + \frac{8}{9} = a_0$

$$\alpha_1 + \frac{8}{9} = 1$$

$$\alpha_1 = 1 - \frac{8}{9} =$$

$$\boxed{\alpha_1 = \frac{1}{9}}$$

Put $n=1$ $\Rightarrow (\alpha_1 + \alpha_2)(-2) + \frac{8}{9} = 2$

$$-2\alpha_1 - 2\alpha_2 + \frac{8}{9} = 2$$

$$-2\left(\frac{1}{9}\right) - 2\alpha_2 + \frac{8}{9} = 2$$

$$-2\alpha_2 = 2 - \frac{2}{3} = \frac{4}{3} \qquad \boxed{\alpha_2 = \frac{-2}{3}}$$

Substituting α_1, α_2 in ④

$$\therefore \quad a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$$

Q) $a_n - 2a_{n-1} + a_{n-2} = 2$ with $a_0 = 25$, $a_1 = 16$

Sol: Given,

$$a_n - 2a_{n-1} + a_{n-2} = 2 \longrightarrow ①$$

It is a non-liner r.r of order 2

The general solution is $a_n = a_n^{(h)} + a_n^{(p)} \longrightarrow ②$

(i) To obtain $a_n^{(h)}$, $f(n) = 0$

$$a_n - 2a_{n-1} + a_{n-2} = 0$$

Ch. Eq: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

$$a_n = (\alpha_1 + \alpha_2 n)\, 1^n$$

ii) To obtain $a_n^{(p)} = d$

$d - 2d + d = 2$

$0 = 2$ ×

$a_n^{(p)} = nd$

$nd - 2(n-1)d + (n-2)d = 2$

$\cancel{nd} - 2nd + 2\cancel{d} + \cancel{n}d - 2\cancel{d} = 2$

Assume $\quad a_n^{(p)} = n^2 d$

$$n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$$

$$n^2 d - 2(n^2+1-2n)d + (n^2+4-4n)d = 2$$

$$\cancel{n^2 d} - \cancel{2n^2 d} + 2d + \cancel{4nd} + \cancel{n^2 d} + 4d - \cancel{4nd} = 2$$

$$2d = 2$$

$$\boxed{d = 1}$$

(iii) $\quad a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = (\alpha_1 + \alpha_2 n) + n^2$$

$n = 0 \Rightarrow \quad (\alpha_1 + 0) + 0^2 = 25$

$$\alpha_1 = 25$$

$n = 1 \Rightarrow \quad (\alpha_1 + \alpha_2) + 1 = a_1$

$$25 + \alpha_2 + 1 = 16$$

$$\alpha_2 = 16 - 25 - 1$$

$$\alpha_2 = -10$$

$$a_n = (25 - 10n) + n^2$$

$$\therefore a_n = n^2 - 10n + 25$$

Q) Solve the following recurrence relation

$$a_n = 3a_{n-1} + 2^n \quad n \geq 1 \quad a_0 = 3$$

Sol: Given,

$$a_n = 3a_{n-1} + 2^n$$

$$\Rightarrow a_n - 3a_{n-1} = 2^n \longrightarrow ①$$

The solution for the equation is:

$$a_n = a_n^{(h)} + a_n^{(p)} \longrightarrow ②$$

Step 1: The associated homogeneous relation $a_n^{(h)}$

$$a_n - 3a_{n-1} = 0$$

Ch: Eqn $\Rightarrow$ $r - 3 = 0$

$$\boxed{r = 3}$$

$$a_n = \alpha_1 3^n \longrightarrow ③$$

Step 2: To obtain particular solution of the given recurrence relation is

$a_n^{(p)} = 2^n \longrightarrow ④$ and 2 is not the characteristic root

Let Particular sol^n be $a_n^{(P)} = d\,2^n \longrightarrow ⑤$

sub. (5) in (1)

$\Rightarrow \quad d\,2^n - 3d\,2^{n-1} = 2^n$

$d\,\cancel{2^n}\left(1 - \frac{3}{2}\right) = \cancel{2^n}$

$d\left(\frac{-1}{3}\right) = 1$

$d = -3$

$d\,\cancel{2^{n-1}}(2-3) = \cancel{2^n}\,2$

$\boxed{d = -2}$ in (5)

$a_n^{(p)} = (-2)\,2^n$

Put values of $a_n^{(h)}$ & $a_n^{(p)}$ in (2)

(2) $\Rightarrow \quad a_n = \alpha\,3^n + (-2)\,2^n$

Put $n = 0 \Rightarrow$

$\alpha\,3^0 + (-2)\,2^0 = a_0$

$\alpha - 2 = 3$

$\boxed{\alpha = 5}$

$\therefore \; a_n = 5(3^n) - 2^{n+1}$

Q) Solve the following recurrence relation

$$a_n = 2a_{n-1} + 2^n \quad n \geq 1 \quad a_0 = 2$$

Sol: Given,

$$a_n = 2a_{n-1} + 2^n$$

$$\Rightarrow a_n - 2a_{n-1} = 2^n \longrightarrow (1)$$

It is a non-linear RR with order 1

General soln is $a_n = a_n^{(h)} + a_n^{(P)} \longrightarrow (2)$

Step 1: Obtain $a_n^{(h)}$ by $f(n) = 0$

$$a_n - 2a_{n-1} = 0$$

ch. eqn is: $r - 2 = 0$

$$\boxed{r = 2}$$

$$a_n^{(h)} = \alpha 2^n \longrightarrow (3)$$

Step 2: Obtain $a_n^{(P)}$ by $a_n^{(P)} = 2^n$

$a_n^{(P)} = d 2^n$ Characteristic root is equal

put $a_n^{(P)} = nd 2^n$ in (1)

$$(1) \Rightarrow nd2^n - 2(n-1)d\, 2^{n-1} = 2^n$$

$$\cancel{2^n}(nd - (n-1)d) = \cancel{2^n}$$

$\cancel{nd} - \cancel{nd} + d = 1$

$\boxed{d = 1}$

$a_n^{(p)} = n2^n$

Put $a_n^{(h)}$ & $a_n^{(p)}$ values in (2)

(2) $\Rightarrow$ $a_n = \alpha 2^n + n2^n$

$n = 0$ | $\alpha 2^0 + 0 = a_0$

$\alpha = 2$

$\therefore$ $a_n = 2^{n+1} + n2^n$ (or) $2^n(2+n)$

Q) $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

Sol: $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)}$: $r^2 + 5r + 6 = 0$

$r = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2} = -2, -3$

$a_n^{(h)} = \alpha_1(-2)^n + \alpha_2(-3)^n \longrightarrow (1)$

$$a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$$

$$a_n^{(P_1)} = 3 \text{ (const.)} \qquad a_n^{(P_2)} = n^2$$

$$a_n^{(P_1)} = d \text{ in } 1 \qquad d_0 + d_1 k + d_2 k^2 = n^2$$

$$d + 5d + 6d = 3$$

$$12d = 3$$

$$d = \frac{1}{4}$$

$$a_n^{(P_2)} = n^2$$

$$d_0 + d_1 k + d_2 k^2 = n^2$$

$$5a_{n-1}$$

$$d_0 + d_1 n + d_2 n^2 + 5\left[d_0 + d_1(n-1) + d_2(n-1)^2\right] + 6\left[d_0 + d_1(n-2) + d_2(n-2)^2\right] = n^2$$

$$d_0 + d_1 n + d_2 n^2 + 5\left[d_0 + d_1 n - d_1 + d_2 n^2 + d_2 - 2nd_2\right] + 6\left[d_0 + d_1 n - 2d_1 + n^2 d_2 + 4d_2 - 4nd_2\right] = n^2$$

$$d_0 + nd_1 + n^2 d_2 + 5d_0 + 5nd_1 - 5d_1 + 5n^2 d_2 + 5d_2 - 10nd_2 + 6d_0 + 6nd_1 - 12d_1 + 6n^2 d_2 + 24d_2 - 24nd_2 = n^2$$

$$12d_0 + 8nd_1 + 12n^2 d_2 + 17d_1 + 29d_2 - 34nd_2 = n^2$$

$$12n^2 d_2 - 34nd_2 + \overset{12}{8}nd_1 + 12d_0 + 17d_1 + 29d_2 = n^2$$

$$(12d_2 - 1)n^2 + n(12d_1 - 34d_2) + (12d_0 - 17d_1 + 29d_2) = 0$$

$$12d_2 - 1 = 0 \qquad\qquad 12d_1 = 34d_2 \qquad\qquad 12d_0 - 17d_1 + 29d_2 = 0$$

$$\boxed{d_2 = \frac{1}{12}} \qquad\qquad d_1 = \frac{34}{12}\left(\frac{1}{12}\right)$$

$$\boxed{d_1 = \frac{17}{72}}$$

$$d_0 = \frac{1}{12}\left(17\left(\frac{17}{72}\right) - 29\left(\frac{1}{12}\right)\right)$$

$$d_0 = \frac{1}{12}\left(\frac{289}{72} - \frac{29}{12}\right)$$

Compare coefficients

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$$\boxed{d_0 = \frac{115}{864}}$$

$$a_n^{(P_2)} = \left(\frac{115}{864} + \frac{17}{72}n + \frac{1}{12}n^2\right) \times \frac{1}{4}$$

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = \alpha_1(-2)^n + \alpha_2(-3)^n + \frac{1}{4}\left(\frac{115}{864} + \frac{17n}{72} + \frac{n^2}{12}\right)$$

1) Solve $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$
$a_1 = 7$

Sol: Given,

$$a_n - a_{n-1} - 2a_{n-2} = 0$$

2nd Degree Linear Recc. Relation.

Characteristic Eqn: $r^2 - r - 2 = 0$

$$r = \frac{1 \pm \sqrt{1-4(1)(-2)}}{2}$$

$$r = \frac{1 \pm 3}{2} = 2, -1$$

$$a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \longrightarrow ①$$

Put $n = 0 \Rightarrow \alpha_1 (2)^0 + \alpha_2 (-1)^0 = a_0$

$\alpha_1 + \alpha_2 = 2 \longrightarrow ⓐ$ (Given $a_0 = 2$)

Put $n = 1 \Rightarrow \alpha_1 (2)^1 + \alpha_2 (-1)^1 = a_1$

$2\alpha_1 - \alpha_2 = 7 \longrightarrow ⓑ$ (Given $a_1 = 7$)

$\alpha_2 = 2\alpha_1 - 7$ in ⓐ

ⓐ $\Rightarrow \alpha_1 + 2\alpha_1 - 7 = 2$

$3\alpha_1 = 9$

$\boxed{\alpha_1 = 3}$ in ⓑ

ⓑ $\Rightarrow 2(3) - \alpha_2 = 7 \Rightarrow \boxed{\alpha_2 = -1}$

Put α_1, α_2 in ① $\Rightarrow a_n = 3(2)^n - (-1)^n$ is req. soln

2) Solve $a_n = 2a_{n-1} + 3 * 2^n$

Sol: Given, $a_n - 2a_{n-1} = 3 * 2^n \longrightarrow (1)$

1st Order Non-Linear Recc. Relation

General solⁿ is

$$a_n = a_n^{(h)} + a_n^{(P)} \longrightarrow (2)$$

To find $a_n^{(h)}$

$$a_n - 2a_{n-1} = 0$$

Ch. Eqn. $\quad r - 2 = 0$

$$\boxed{r = 2}$$

$$a_n^{(h)} = \alpha (2)^n \longrightarrow (3)$$

To find $a_n^{(P)}$

$$a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$$

$a_n^{(P_1)} = 3$

Sol: $a_n^{(P_1)} = d$ in (1)

(1) $\Rightarrow$ $d - 2d = 3$

$$\boxed{d = -3}$$

$$a_n^{(P_1)} = -3$$

$a_n^{(P_2)} = 2^n$

$a_n^{(P_2)} = nd2^n$ (2 is a characteristic root)

in (1)

(1) $\Rightarrow$ $nd2^n - 2d(n-1)2^{n-1} = 2^n$

$nd2^n - d(n-1)2^n = 2^n$

$\cancel{2^n}d[n-(n-1)] = \cancel{2^n}$

$$\boxed{d = 1}$$

$$a_n^{(P_2)} = n2^n$$

$$a_n^{(P)} = -3n\,2^n$$

Put values of $a_n^{(H)}$ & $a_n^{(P)}$ in ②

$$② \Rightarrow \quad a_n = \alpha(2^n) - 3n\,2^n$$

$\therefore \; a_n = 2^n[\alpha - 3n]$ is the required solution.

3) Solve $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 3$, $a_1 = 5$.

Sol: Given,

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

II Order Linear Recurrence relation

Ch. Eqn is

$$r^2 - 7r + 10 = 0$$

$$(r-5)(r-2) = 0$$

$$\boxed{r = 5, 2}$$

The ch. roots are real & distinct. General soln is

$$a_n = \alpha_1 5^n + \alpha_2 2^n \longrightarrow ①$$

Put $n = 0$ in ① $\Rightarrow$ $\alpha_1 5^0 + \alpha_2 5^0 = a_0$

$$\alpha_1 + \alpha_2 = 3 \longrightarrow ⓐ$$

$$\alpha_2 = 3 - \alpha_1 \text{ in } ⓑ$$

Put $n = 1$ in ① $\Rightarrow$ $\alpha_1 5^1 + \alpha_2 2^1 = a_1$

$$5\alpha_1 + 2\alpha_2 = 5 \longrightarrow ⓑ$$

$$5\alpha_1 + 2(3 - \alpha_1) = 5$$

$$5\alpha_1 + 6 - 2\alpha_1 = 5$$

$$3\alpha_1 + 6 = 5$$

$$3\alpha_1 = -1$$

$$\boxed{\alpha_1 = \frac{-1}{3}}$$

$$\alpha_2 = 3 + \frac{1}{3}$$

$$\boxed{\alpha_2 = \frac{10}{3}}$$

Put α_1, α_2 in ①

① $\Rightarrow a_n = \frac{-1}{3}(5^n) + \frac{10}{3}(2^n)$ is the required solution.

4) $a_n = a_{n-1} + 3^n$

Sol: Given,

$$a_n - a_{n-1} = 3^n \rightarrow ①$$

I-Order Non-Linear Recc. Relation.

~~Ch. Eq. is~~ ~~r-1~~ General sol^n is of $a_n = a_n^{(H)} + a_n^{(P)} \rightarrow ②$

To find $a_n^{(H)}$

$$a_n - a_{n-1} = 0$$

Ch. Eq. is $r - 1 = 0$

$$\boxed{r = 1}$$

$$a_n^{(H)} = \alpha(1)^n = \alpha \rightarrow ③$$

To find $a_n^{(P)}$

$a_n^{(P)} = 3^n$ is of the form r^n

Particular sol^n is $a_n^{(P)} = d\,3^n$ in ①

① $\Rightarrow d\,3^n - d\,3^{n-1} = 3^n$

$$d\cancel{3^n}\left[1 - \frac{1}{3}\right] \cancel{3^n}$$

$$d\left[\frac{2}{3}\right] = 1$$

$\boxed{d = \frac{3}{2}}$ $\qquad a_n^{(P)} = \frac{3}{2}(3^n) \rightarrow ④$

values of $a_n^{(H)}$ & $a_n^{(P)}$ in ②

from ③,④ $\Rightarrow$ $a_n = \alpha + \frac{1}{2}(3^{n+1})$

5.) Solve $a_n - 3a_{n-1} + 2a_{n-2} = 0$ for $n \geq 2$

Sol: Given,

$$a_n - 3a_{n-1} + 2a_{n-2} = 0$$

II - Order Linear Recc. Relation

Ch. Eqn. is $r^2 - 3r + 2 = 0$

$$(r-1)(r-2) = 0$$

$$\boxed{r = 1, 2}$$

The characteristic roots are real & distinct

$$\therefore \quad a_n = \alpha_1 1^n + \alpha_2 (2^n)$$

$$a_n = \alpha_1 + \alpha_2 (2^n)$$

6.) Solve $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$ for $n \geq 2$ with $a_0 = 3$, $a_1 = 7$

Sol: Given,

$$a_n - 6a_{n-1} + 8a_{n-2} = 3^n \longrightarrow ①$$

It is a II - Order Non Linear Recurrence Relation

The general sol^n is of the form: $a_n = a_n^{(H)} + a_n^{(P)}$

To find $a_n^{(H)}$

$a_n^{(H)}$ $a_n - 6a_{n-1} + 8a_{n-2} = 0$

Characteristic Equation is $r^2 - 6r + 8 = 0$

$(r-4)(r-2) = 0$

$\boxed{r = 2, 4}$

The ch. roots are real & distinct

$a_n^{(H)} = \alpha_1 (2^n) + \alpha_2 (4^n) \longrightarrow (2)$

To find $a_n^{(P)}$

$a_n^{(P)} = 3^n$ is of the form $a_n = r^n$

Sol^n is $a_n^{(P)} = d\,3^n$ in (1)

(1) $\Rightarrow$ $d\,3^n - 6d\,3^{n-1} + 8d\,3^{n-2} = 3^n$

$d\,\cancel{3^n}\left[1 - \frac{\overset{2}{\cancel{6}}}{3} + \frac{8}{9}\right] = \cancel{3^n}$

$d\left[\frac{8}{9} - 1\right] = 1$

$d\left[\frac{-1}{9}\right] = 1 \quad \Rightarrow \quad \boxed{d = -9}$

$a_n^{(P)} = -9(3^n) \longrightarrow (3)$

Put (2) & (3) values in a_n

$\Rightarrow a_n = \alpha_1 (2^n) + \alpha_2 (4^n) - 9(3^n) \longrightarrow (4)$

$\alpha_1 (2^n) + \alpha_2 (4^n) - 3^{n+2}$

$n=0$ in (4) $\Rightarrow \alpha_1(2^0) + \alpha_2(4^0) - 9(3^0) = a_0$

$$\alpha_1 + \alpha_2 - 9 = 3$$

$$\alpha_2 = 12 - \alpha_1$$

Put $n=1$ in (4) $\Rightarrow \alpha_1(2^1) + \alpha_2(4^1) - 9(3^1) = a_1$

$$2\alpha_1 + 4\alpha_2 - 27 = 7$$

$$2\alpha_1 + 4(12 - \alpha_1) = 34$$

$$2\alpha_1 + 48 - 4\alpha_1 = 34$$

$$-\cancel{2}\alpha_1 = \cancel{-}14$$

$$\boxed{\alpha_1 = 7}$$

$$\alpha_2 = 12 - \alpha_1$$
$$= 12 - 7$$
$$\boxed{\alpha_2 = 5}$$

Put α_1, α_2 in (4) $\Rightarrow a_n = 7(2^n) + 5(4^n) - 3^{n+2}$

is the required solution.

7.) Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 2$

Sol: Given,

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

It is a II Order Homogeneous Recurrence Relation

Characteristic Eq. is $r^2 - 5r + 6 = 0$

$$(r-2)(r-3) = 0$$

$$\boxed{r = 2, 3}$$

The chr. roots are real & distinct

General sol^n is

$$a_n = \alpha_1(2^n) + \alpha_2(3^n) \longrightarrow (1)$$

Put $n=0$ in (1) $\Rightarrow$ $\alpha_1(2^0) + \alpha_2(3^0) = a_0$

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_2 = 1 - \alpha_1$$

Put $n=1$ in (1) $\Rightarrow$ $\alpha_1(2^1) + \alpha_2(3^1) = a_1$

$$2\alpha_1 + 3\alpha_2 = -2$$

$$2\alpha_1 + 3(1-\alpha_1) = -2$$

$$2\alpha_1 + 3 - 3\alpha_1 = -2$$

$$+\boxed{\alpha_1 = 5}$$

$$\alpha_2 = 1 - \alpha_1$$

$$= 1 - 5$$

$$\boxed{\alpha_2 = -4}$$

Put the values of α_1, α_2 in (1)

$\Rightarrow$ $a_n = 5(2^n) - 4(3^n)$ is the required solution.

8.) Solve $a_n - 6a_{n-1} + 8a_{n-2} = n4^n$ for $n \geq 2$

Sol: Given,

$$a_n - 6a_{n-1} + 8a_{n-2} = n4^n \longrightarrow (1)$$

It is II-Order Non Linear Recc. Relation.

The general sol^n is $a_n = a_n^{(H)} + a_n^{(P)} \longrightarrow (2)$

d $a_n^{(H)}$

$$a_n - 6a_{n-1} + 8a_{n-2} = 0$$

Characteristic Equation is $r^2 - 6r + 8 = 0$

$$(r-4)(r-2) = 0$$

$$\boxed{r = 2, 4}$$

The characteristic roots are real & distinct

$$a_n^{(H)} = \alpha_1(2^n) + \alpha_2(4^n) \longrightarrow (3)$$

To find $a_n^{(P)}$ $\quad a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$

$$a_n^{(P_1)} = n$$

It is linear polynomial

$a_n^{(P_1)} = d_0 + d_1 n$ in (1)

$$d_0 + d_1 n - 6[d_0 + d_1(n-1)] + 8[d_0 + d_1(n-2)] = n$$

(1) $\Rightarrow$ $d_0 + d_1 n - 6d_0 - 6nd_1 + 6d_1 + 8d_0 + 8nd_1 - 16d_1 = n$

$$d_0 + d_1 n - 6d_0 - 6nd_1 + 6d_1$$

$$3nd_1 - 10d_1 + 3d_0 = n$$

Equating coeff. of n $\Rightarrow$ $3d_1 = 1$

$$\boxed{d_1 = \frac{1}{3}}$$

Constants $\Rightarrow$ $3d_0 - 10d_1 = 0$

$$3d_0 - 10\left(\frac{1}{3}\right) = 0$$

$$\boxed{d_0 = \frac{10}{9}} \qquad a_n^{(P_1)} = \frac{10}{9} + \frac{1}{3}n$$

$a_n^{(P_2)} = 4^n$ which is a characteristic root

$\Rightarrow a_n^{(P_2)} = dn4^n$ in ①

$$dn4^n - 6d(n-1)4^{n-1} + 8d(n-2)4^{n-2} = 4^n$$

$$d\cancel{4^n}\left[n - \overset{3}{\cancel{6}}(n-1)\cdot\frac{1}{\underset{2}{\cancel{4}}} + 8(n-2)\frac{1}{16}\right] = \cancel{4^n}$$

$$d\left[n - \frac{3}{2}(n-1) + \frac{1}{2}(n-2)\right] = 1$$

$$d\left[\cancel{2n} - \cancel{3n} - 3 + \cancel{n} - 2\right] = 2$$

$$d[-5] = 2 \quad \Rightarrow \quad \boxed{d = \frac{-2}{5}}$$

$$a_n^{(P_2)} = \frac{-2n}{5}(4^n)$$

$$a_n^{(P)} = \left(\frac{10}{9} + \frac{n}{3}\right)\left(\frac{-2n}{5}(4^n)\right) \longrightarrow ④$$

Put values of ③ & ④ in ②

$$② \Rightarrow a_n = \alpha_1(2^n) + \alpha_2(4^n) + \left(\frac{10}{9} + \frac{n}{3}\right)\left(\frac{-2n}{5}4^n\right)$$

is the required solution.

Solve $a_{n+1} - 10a_n + 9a_{n-1} = 5 * 9^n$, $n \geq 1$ with $a_0 = 1$, $a_1 = 4$

Sol: Given,

$$a_{n+1} - 10a_n + 9a_{n-1} = 5 * 9^n \longrightarrow ①$$

It is a II Order Non-Linear/ ~~Homogeneous~~ Recc. Relation.

The sol^n is of the form

$$a_n = a_n^{(H)} + a_n^{(P)} \longrightarrow ②$$

To find $a_n^{(H)}$:

The associated linear recc. rel^n is

$$a_{n+1} - 10a_n + 9a_{n-1} = 0$$

The ch. eqn. is $r^2 - 10r + 9 = 0$

$$(r-9)(r-1) = 0$$

$$\boxed{r = 1, 9}$$

The char. roots are real and distinct

$$a_n^{(H)} = \alpha_1 (1)^n + \alpha_2 (9^n)$$

$$a_n^{(H)} = \alpha_1 + \alpha_2 (9^n) \longrightarrow ③$$

To find $a_n^{(P)}$: $a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$

$a_n^{(P_1)} = 5$ (Constant)

Sol is $a_n^{(P_1)} = d$ in ①

① ⇒ $d - 10d + 9d = 5$

$0 = 5$

Now assume $a_n^{(P_1)} = nd$

$① \Rightarrow \quad (n+1)d - 10dn + 9d(n-1) = 5$

$\cancel{nd} + d - 1\cancel{0}nd + 9\cancel{n}d - 9 = 5$

$d - 9 = 5$

$\boxed{d = 14}$

$a_n^{(P_1)} = 14$

$a_n^{(P_2)} = 9^n$ which is a characteristic root

Assume $a_n^{(P_2)} = dn9^n$ in ①

$① \Rightarrow d(n+1)9^{n+1} - 10nd9^n + 9(n-1)d9^{n-1} = 9^n$

$d\cancel{9^n}[9(n+1) - 10n + (n-1)] = \cancel{9^n}$

$d[9\cancel{n} + 9 - 1\cancel{0}n + \cancel{n} - 1] = 1$

$8d = 1 \quad \Rightarrow \boxed{d = \frac{1}{8}}$

$a_n^{(P_2)} = \frac{n}{8}9^n$

$a_n^{(P)} = \overset{7}{\cancel{14}} * \frac{n}{\underset{4}{\cancel{8}}}9^n = \frac{7}{4}n9^n \longrightarrow ④$

③, ④ in ②

$\therefore \quad a_n = \alpha_1 + \alpha_2(9^n) + \frac{7}{4}n9^n \longrightarrow ⑤$

$1 = 0$ in ⑤ $\Rightarrow \quad \alpha_1 + \alpha_2(9^0) + \frac{7}{4} \times 0 \times 9^n = a_0$

$\alpha_1 + \alpha_2 = 1 \qquad \Rightarrow \alpha_2 = (1 - \alpha_1)$

$n = 1$ in ⑤ $\Rightarrow \quad \alpha_1 + 9\alpha_2 + \frac{7}{4}(9) = a_1$

$\alpha_1 + 9\alpha_2 + \frac{63}{4} = 4$

$\alpha_1 + 9 - 9\alpha_1 + \frac{63}{4} = 4$

$$-8\alpha_1 = 4 - \frac{99}{4}$$

$$-8\alpha_1 = \frac{16-99}{4}$$

$$\boxed{\alpha_1 = \frac{83}{32}}$$

$$\alpha_2 = 1 - \frac{83}{32}$$

$$\boxed{\alpha_2 = \frac{-51}{32}}$$

Put α_1, α_2 in (5)

$$\therefore \quad a_n = \frac{83}{32} - \frac{51}{32}(9^n) + \frac{7n}{4} 9^n$$

10.) $a_{n+2} - 6a_{n+1} + 9a_n = 10 * 3^n, \quad n \geq 0$

Sol: Given,

$$a_{n+2} - 6a_{n+1} + 9a_n = 10 * 3^n \longrightarrow (1)$$

It is II Order Non-Linear Recc. Relation

The general sol^n is:

$$a_n = a_n^{(H)} + a_n^{(P)}$$

To find $a_n^{(H)}$:

The associated linear recc. relation is

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

Ch. Eqn: $r^2 - 6r + 9 = 0$

$$(r-3)(r-3) = 0$$

$$\boxed{r = 3, 3}$$

The characteristic roots are real and equal

$$a_n^{(H)} = (\alpha_1 + n\alpha_2) 3^n \longrightarrow (2)$$

To find $a_n^{(P)}$

$$a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$$

$$a_n^{(P_1)} = 10$$

R.H.S is constant

assume $a_n^{(P_1)} = d$ in ①

$$① \Rightarrow d - 6d + 9d = 10$$

$$\cancel{10d}\ 4d = 10$$

$$\boxed{d = \frac{10}{4} = \frac{5}{2}}$$

$$a_n^{(P_1)} = \frac{5}{2}$$

$$a_n^{(P_2)} = 3^n$$

It is a double characteristic root

Assume $a_n^{(P_2)} = n^2 d\, 3^n$ in ①

$$① \Rightarrow (n+2)^2 d\, 3^{n+2} - 6(n+1)^2 d\, 3^{n+1} + 9n^2 d\, 3^n = 3^n$$

$$(n^2+4+4n)\, d\, 3^{n+2} - 6(n^2+1+2n)\, d\, 3^{n+1} + 9n^2 d\, 3^n = 3^n$$

$$d\,\cancel{3^n}\left[(n^2+4+4n)9 - 6\times 3(n^2+1+2n) + 9n^2\right] = \cancel{3^n}$$

$$d\left[\cancel{9n^2} + 36 + \cancel{36n} - \cancel{18n^2} - 18 - \cancel{36n} + \cancel{9n^2}\right] = 1$$

$$d\,[18] = 1 \quad \Rightarrow \quad \boxed{d = \frac{1}{18}}$$

$$a_n^{(P_2)} = \frac{n^2}{18} 3^n$$

$$a_n^{(P)} = \frac{5n^2}{36} 3^n \longrightarrow ③$$

Put values of ② & ③ in $a_n = a_n^{(H)} + a_n^{(P)}$

$$\therefore a_n = (\alpha_1 + n\alpha_2)3^n + \frac{5n^2}{36} 3^n$$

$a_n = 3^n\left[(\alpha_1 + n\alpha_2 + \frac{5n^2}{36}\right]$ is the required solⁿ.

ie $a_n - 4a_{n-1} + 4a_{n-2} = 0$ $n \geq 2$ with $a_0 = 5/2$
$a_1 = 8$

Sol: Given,

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

It is a II Order Linear Recurrence Relation

Characteristic Eqn. is $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$$\boxed{r = 2, 2}$$

The ch. roots are real & equal, the general soln is:

$$a_n = (\alpha_1 + n\alpha_2)2^n \rightarrow ①$$

Put $n=0$ in ① $\Rightarrow$ $(\alpha_1 + 0)\,2^0 = a_0$

$$\boxed{\alpha_1 = \frac{5}{2}} \qquad (\text{Given } a_0 = 5/2)$$

Put $n=1$ in ① $\Rightarrow$ $(\alpha_1 + \alpha_2)2 = a_1$

$$2\alpha_1 + 2\alpha_2 = 8$$

$$2\left(\frac{5}{2}\right) + 2\alpha_2 = 8$$

$$2\alpha_2 = 3$$

$$\boxed{\alpha_2 = \frac{3}{2}}$$

Put α_1, α_2 in ① $\Rightarrow$ $a_n = \left(\frac{5}{2} + \frac{3n}{2}\right)2^n$

$a_n = (5+3n)\,2^{n-1}$ is required solⁿ.

12) Solve $a_n - 6a_{n-1} + 8a_{n-2} = 9$ $n \geq 2$ $a_0 = 10$

$a_1 = 25$

Sol: Given,

$$a_n - 6a_{n-1} + 8a_{n-2} = 9 \rightarrow ①$$

It is a II Order Non-linear recurrence relation

General solⁿ is given by $a_n = a_n^{(H)} + a_n^{(P)} \rightarrow ②$

To find $a_n^{(H)}$:

The associated linear eqn. is

$$a_n - 6a_{n-1} + 8a_{n-2} = 0$$

characteristic eqn. $r^2 - 6r + 8 = 0$

$$(r-4)(r-2) = 0$$

$$\boxed{r = 2, 4}$$

The characteristic roots are real & distinct

$$a_n = \alpha_1(2^n) + \alpha_2(4^n) \rightarrow ③$$

To find $a_n^{(P)}$:

$$a_n^{(P)} = 9$$

R.H.S is constant

assume, $a_n^{(P)} = d$ in ①

$$① \Rightarrow d - 6d + 8d = 9$$

$$3d = 9$$

$$\boxed{d = 3}$$

$$a_n^{(P)} = 3 \rightarrow ④$$

alues of 3 & 4 in 2

$$2 \Rightarrow a_n = \alpha_1(2^n) + \alpha_2(4^n) + 3 \longrightarrow 5$$

Put $n=0$ in 5 $\Rightarrow \alpha_1(2^0) + \alpha_2(4^0) + 3 = a_0$

$$\alpha_1 + \alpha_2 + 3 = 10$$

$$\alpha_1 + \alpha_2 = 7 \Rightarrow \alpha_2 = 7 - \alpha_1$$

Put $n=1$ in 5 $\Rightarrow \alpha_1(2^1) + \alpha_2(4^1) + 3 = a_1$

$$2\alpha_1 + 4\alpha_2 + 3 = 25$$

$$2\alpha_1 + 4(7-\alpha_1) + 3 = 25$$

$$2\alpha_1 + 28 - 4\alpha_1 + 3 = 25$$

$$-2\alpha_1 = 25 - 31$$

$$2\alpha_1 = 6 \qquad \alpha_2 = 7 - 3$$

$$\boxed{\alpha_1 = 3} \qquad \boxed{\alpha_2 = 4}$$

Substitute α_1, α_2 in 5

$$5 \Rightarrow a_n = 3(2^n) + 4(4^n) + 3$$

$$a_n = 3(2^n + 1) + 4^{n+1}$$

is the required solution.

13.) Solve $a_n - 7a_{n-1} + 10a_{n-2} = 7 * 3^n \quad n \geq 2$

<u>Sol:</u> Given,

$$a_n - 7a_{n-1} + 10a_{n-2} = 7 * 3^n \longrightarrow ①$$

① is II Order Non Linear Recc. Rel^n

The general sol^n is of the form: $a_n = a_n^{(H)} + a_n^{(P)} \longrightarrow ②$

<u>To find $a_n^{(H)}$:</u>

Write associated linear eqn. of ①

$$\Rightarrow a_n - 7a_{n-1} + 10a_{n-2} = 0$$

Ch. Eqn. $r^2 - 7r + 10 = 0$

$(r-2)(r-5) = 0$

$\boxed{r = 2, 5}$

$$a_n^{(H)} = \alpha_1(2^n) + \alpha_2(5^n) \longrightarrow ③$$

<u>To find $a_n^{(P)}$</u>

$$a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$$

$a_n^{(P_1)} = 7$

R.H.S is constant

Assume $a_n^{(P_1)} = d$ in ①

① $\Rightarrow d - 7d + 10d = 7$

$4d = 7$

$\boxed{d = \frac{7}{4}}$

$a_n^{(P_1)} = \frac{7}{4}$

$a_n^{(P_2)} = 3^n$

R.H.S is of the form r^n

Assume $a_n^{(P_2)} = d3^n$ in ①

$d3^n - 7d3^{n-1} + 10d3^{n-2} = 3^n$

$d\cancel{3^n}\left[1 - 7/3 + \frac{10}{9}\right] = \cancel{3^n}$

$d\left[\frac{9 - 21 + 10}{9}\right] = 1$

$d\left[\frac{-2}{9}\right] = 1 \Rightarrow \boxed{d = \frac{-9}{2}}$

$a_n^{(P_2)} = \frac{-9}{2} 3^n$

$$a_n^{(P)} = \frac{7}{4}\left(\frac{-9}{2} 3^n\right)$$

$$a_n^{(P)} = \frac{-63}{8} 3^n \longrightarrow (4)$$

Put (3), (4) in (2)

$$a_n = \alpha_1(2^n) + \alpha_2(5^n) - \frac{63}{8} 3^n$$ is the required sol^n.

14) Solve $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$ with

$a_0 = 1$, $a_1 = 4$, $a_2 = 8$

Sol: Given, $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$

It is III Order linear recc. rel^n.

Ch. Eqn. $r^3 - 7r^2 + 16r - 12 = 0$

$$\boxed{r = 3, 2, 2}$$

$$a_n^{t} = (\alpha_1 + n\alpha_2) 2^n + \alpha_3(3^n) \longrightarrow (1)$$

$n = 0$ in (1) $\Rightarrow$ $\alpha_1 + \alpha_3 = 1 \longrightarrow$ (a)

$n = 1$ in (1) $\Rightarrow$ $(\alpha_1 + \alpha_2) 2 + \alpha_3(3^1) = a_1$

$2\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4 \longrightarrow$ (b)

$n = 2$ in (1) $\Rightarrow$ $4\alpha_1 + 8\alpha_2 + 9\alpha_3 = 8 \longrightarrow$ (c)

Solving ⓐ, ⓑ, ⓒ

$$\alpha_1 = 5, \quad \alpha_2 = 3, \quad \alpha_3 = -4 \quad \text{in } ①$$

① $\Rightarrow$ $a_n = (5+3n)2^n - 4(3^n)$ is the required solution.

Types of Recurrence Relations:

Two types of Recurrence Relation.

① Linear Recurrence relation / Homogeneous Recurrence relation.

② non linear Recurrence relation / non homogeneous or Inhomogeneous recurrence relations

Note:- if the function $f(n)=0$ then the recurrence relation is said to homogeneous otherwise it is said to be non-homogeneous.

Order of the recurrence relation:

It is the difference b/w highest and lowest subscripts.

Example 1. $a_n + a_{n-1} = 0$.   ~~$n-n+1$~~

$n-(n-1) = n-n+1 = 1^{st}$ order.

2. $a_n + a_{n-1} + a_{n-2} = 0$.

R.R $n-(n-2) \Rightarrow n-n+2 = 2$ 2^{nd} order.

3. $a_n + a_{n-1} + a_{n-2} + \cdots a_{n-n} = 0$.

$n-(n-n) = n-n+n = n^{th}$ order

4. $a_n + 2a_{n-1} = n^2$ — non homogeneous Recurrence relation.

$n-(n-1) = n-\cancel{n}+1 = 1^{st}$ order.

$a_n + 3a_{n-2} = 0$.

$n-(n-2) = n-\cancel{n}+2 = 2^{nd}$ order.

Recurrence Relation

- Iteration method
- Characteristic method.
- generating method.

Example:

Methods of Solving Recurrence Relation

Using characteristic method

Procedure:-

1. write characteristic equation for the given recurrence relation.

2. Find the roots. and let roots are to be $r_1, r_2, r_3 \ldots r_n$.

3. If r_1 and r_2 are real and distinct, then the solution is $\boxed{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$ where α_1, α_2 are arbitrary constants.

4. If r_1 and r_2 are real and equal. then the solution is $\boxed{a_n = (\alpha_1 + \alpha_2 n) r^n}$ where α_1 & α_2 are arbitrary constants.

5. r_1 & r_2 are complex, then the solution is.

$$a_n = r^n(\alpha_1 \cos n\phi + \alpha_2 \cos n\phi)$$ where α_1 & α_2 are arbitrary constants.

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Relations:

Example: find the solution of the recurrence relation

$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5$, with

$a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

Sol: The Given Recurrence Relation is

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0.$$

The characteristic equation is

$$r^3 - 2r^2 - r + 2 = 0$$

1	1	-2	-1	2
	0	1	-1	-2
	1	-1	-2	0

$(r-1)(r^2 - r - 2) = 0.$

$(r-1)(r+1)(r-2) = 0.$

$\Rightarrow r = 1, 2, -1.$

$c_1 + c_2 + c_3 = 3$

$c_1 + 2c_2 - c_3 = 6$

$3c_2 = 9$

$c_2 = 9/3 = 3$

Hence the solution is.

$$a_n = c_1 1^n + c_2 2^n + c_3 (-1)^n.$$

where c_1, c_2, c_3 are arbitrary constants.

Initial conditions are:

$a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

if $a_0 = 3$, $a_0 = c_1 1^0 + c_2 2^0 + c_3 (-1)^0$

$\Rightarrow c_1 + c_2 + c_3 = 3 \Rightarrow c_1 + c_2 + c_3 = 3$ — (1)

when $a_1 = 6$, $a_1 = c_1 1^1 + c_2 2^1 + c_3 (-1)^1$

$c_1 + 2c_2 - c_3 = 6.$ — (2)

If $a_2 = 0$:

$$c_1 + c_2 2^2 + c_3(-1)^2 = 0$$

$$\Rightarrow 4c_2 + c_3 = 0 \quad \text{—— (3)}$$

$c_1 = 6$

$c_2 = -1$

$c_3 = -2$

$a_0 = 3$
$a_1 = 6$
$a_2 = 0$
$6 - 1 - 2 \ldots$
$6 - 4 - 2$

The unique solution is

$$\boxed{a_n = 6(1)^n - 2^n - 2(-1)^n}$$

Example: Solve the recurrence relation ~~a_n~~

$a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$

given that $a_0 = -1$ and $a_1 = 8$

Solution: Given Recurrence Relation is

$$a_n + a_{n-1} - 6a_{n-2} = 0$$

characteristic equation of given recurrence relation

$$r^2 + r - 6 = 0$$

$$r(r+3) - 2(r+3) = 0$$

$$(r-2)(r+3) = 0$$

$$\therefore r = 2, -3$$

roots are real and distinct

Case 1: The general solution is

$a_n = c_1(2)^n + c_2(-3)^n$ $\quad \therefore a_n = A\cdot(k_1)^n + B\cdot(k_2)^n$

c_1 & c_2 are constants.

The initial conditions are

$$a_0 = -1, \quad a_1 = 8$$

Now take $a_0 = -1$ i.e. $n = 0$.

$$a_0 = c_1 \cdot (2)^0 + c_2(-3)^0$$

$$c_1 + c_2 = -1 \quad \text{——— (1)}$$

$$a_1 = c_1 \cdot 2^1 + c_2(-3)^1$$

$$2c_1 - 3c_2 = 8 \quad \text{——— (2)}$$

from (1) & (2)

$$\begin{array}{r} \cancel{2c_1} + 2c_2 = -2 \\ \cancel{2c_1} - 3c_2 = 8 \\ \hline 5c_2 = -10 \end{array} \Rightarrow c_2 = -10/5 = -2.$$

$$c_1 - 2 = -1$$

$$c_1 = -1 + 2 \Rightarrow c_1 = 1.$$

$\therefore c_1 = 1, \; c_2 = -2$

Substitute in general solution.

$$a_n = 1 \cdot 2^n + (-2)(-3)^n$$

$$a_n = (2)^n - 2(-3)^n$$

Example: Solve the recurrence relation.

$a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$.

given that $a_0 = 5$ and $a_1 = 12$.

Sol: Given recurrence relation is.

$a_n - 6a_{n-1} + 9a_{n-2} = 0$ ——— (1)

Characteristic equation for (1) is.

$r^2 - 6r + 9 = 0$ ——— (2)

roots for eq. (2)

~~$r(r+3) + 3(r-3) = 0$~~ $r(r-3) - 3(r-3) = 0$

~~$(r+3)(r-3) = 0$~~. $(r-3)(r-3) = 0$.

$r = 3, 3$. $r_1 = 3$, $r_2 = 3$.

Now the general solution of given recurrence relation is.

$a_n = c_1 \cdot (r_1)^n + c_2 (r_2)^n$.

$a_n = c_1 \cdot (-3)^n + c_2 (3)^n$. ——— (3)

$\therefore c_1$ and c_2 are the constants,

Initial conditions are.

$a_0 = 5$ & $a_1 = 12$.

if $a_0 = 5$ when $n = 0$.

Substitute in eq. (3)

$a_0 = c_1 \cdot (-3)^0 + c_2 (3)^0$

$c_1 + c_2 = 5$ ——— (4)

if a_1

If $a_1 = 12$, where $n = 1$.

$a_1 = c_1(+3)^1 + c_2(3)^1$.

$+3c_1 + 3c_2 = 12$. —— (5)

$\downarrow$ $c_1 + c_2 = 4$. —— (5)

from (4) & (5).

$3c_1 + 3c_2 = 15$

$\underline{-3c_1 \pm 3c_2 = -12}$

$6c_2 = 27$

$c_2 = 27/6 =$

$c_1 + c_2 = 5$

$c_1 + c_2 = 4$

Roots are equal and real. The general solution for this is,

$a_n = (c_1 + c_2 n) \cdot r^n$.

$= (c_1 + c_2 n) \cdot 3^n$. —— (3)

$a_0 = 5, \; a_1 = 12$.

$a_0 = (c_1 + c_2 \cdot 0) \cdot 3^0$.

$c_1 + 0 = 5 \Rightarrow c_1 = 5$.

$a_1 = (c_1 + c_2 \cdot 1) \cdot 3^1$.

$3c_1 + 3c_2 = 12$.

$c_1 + c_2 = 12/3 = 4$

$5 + c_2 = 4 \Rightarrow c_2 = 4 - 5 = -1$.

Substitute the constant values in the equation

$a_n = (5 + (-1) \cdot n) \cdot 3^n$.

$$\boxed{a_n = (5 - n) \cdot 3^n}$$

$\underline{a_0 = 5}$

$n = 0$.

Check the equation is correct or not.

$a_0 = 5$.

$n = 0$

$a_0 = (5 - 0) \cdot 3^0 = 5 \cdot 1 = 5$.

$a_1 = 12$.

$n = 1$.

$a_1 = (5 - 1) \cdot 3^1 = 4 \cdot 3 = 12$.

Final solution is correct

$$\boxed{a_n = (5 - n) \cdot 3^n}$$

——— 0 ———

Solutions of Non-homogeneous Recurrence Relation or Inhomogeneous Recurrence Relation:

If Given Recurrence Relation is a second order non-homogeneous recurrence relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} + f(n) \quad \text{--- (1)}$$

∴ The general solution of relation is:

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}} \quad \text{--- (2)}$$

To obtain $a_n^{(h)}$, put $f(n) = 0$ in equation (1)

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \longrightarrow (3)$$

Step 1:-
We obtain the homogeneous solution:

First we write the associated homogeneous recurrence relation, namely

$$f(n) = 0.$$

i.e $a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} \cdots + c_k a_{n-k} = 0$

Then we find the general solution, which is called the homogeneous solution.

Step 2:-
We obtain the particular solution. There is no general procedure for finding the particular solution of a Recurrence Relation.

However, if $f(n)$ has any one of the following forms.

(i) polynomial in 'n'

(ii) a constant

(iii) powers of constant.

Particular solution for given f(n):

S.NO	f(n)	form of particular solution
1	A constant, C.	A constant, d
2	A linear function $c_0+c_1 n$. N	A linear function $d_0+d_1 k$. $d_0+d_1 k$.
3	n^2	$d_0+d_1 k+d_2 k^2$.
4.	An mth degree polynomial $c_0+c_1 n+c_2 n^2+\dots+c_m n^m$	An mth degree polynomial $d_0+d_1 k+d_2 k^2+\dots+d_m k^m$
5.	$r^n, r \in R$	$d r^n$.

→ ~~Solve the following recurrence relation.~~

~~$a_n+4a_{n-1}+4a_{n-2}=8^n$~~

Step3: The general solution of the recurrence relation is the sum of the homogeneous and particular solution.

Step4: If no initial conditions are given, then step3 will give the solution.

If 'n' initial conditions are given, then we get 'n' equations with 'n' unknowns.

Solution.

Q:- Solve the following Recurrence Relation.

$a_n + 4a_{n-1} + 4a_{n-2} = 8$ for $n \geq 2$ with

$a_0 = 1, \quad a_1 = 2.$

Sol:- Given Recurrence Relation is

$$a_n + 4a_{n-1} + 4a_{n-2} = 8, \text{ for } n \geq 2 \quad \text{—— (1)}$$

Given Recurrence Relation is a second order non-homogeneous recurrence relation.

$\therefore$ The general solution of relation (1) is

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}} \quad \text{—— (2)}.$$

To obtain $a_n^{(h)}$, put $f(n) = 0$ in equation (1)

$$a_n + 4a_{n-1} + 4a_{n-2} = 0 \quad \text{—— (3)}.$$

The characteristic ~~roots~~ equation of equation (3) is.

$$r^2 + 4r + 4 = 0 \quad \text{—— (4)}$$

Characteristic roots of (4) is

$$r^2 + 2r + 2r + 4 = 0$$

$$r(r+2) + 2(r+2) = 0$$

$$r+2 = 0 \quad \& \quad r+2 = 0$$

$$r = -2, -2.$$

Here The roots are equal and real.

The general solution of $a_n^{(h)}$ is.

$$\boxed{a_n^{(h)} = (A + Bn) r^n}.$$

$$a_n^{(h)} = (A + Bn)(-2)^n.$$

obtain the $a_n^{(P)}$ will be calculated.

$$a_n^{(P)} = A_0 \quad \text{---} (8)$$

equation (4) & (5) are substituted in eq (3).

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = (A + Bn)(-2)^n + A_0 \quad \text{---} (6)$$

for obtaining A_0 value, eq (5) is substituted in eq (1)

$$a_n + 4a_{n-1} + 4a_{n-2} = 8$$

$$A_0 + 4A_0 + 4A_0 = 8$$

$$9A_0 = 8$$

$$\boxed{A_0 = 8/9}$$

$A_0 + 4A_0 + 4A_0 = 8$
$A_0 \cdot 5 + 4A_0 = 8$
$9A_0 = 8$
$A_0 = 8/9$

$d + 4(d-1) + 4(d-2) = 8$
$d + 4d - 4 + 4d - 8$
$9d - 12 = 8$
$9d = 8 + 12$
$d = \frac{20}{9}$

Now substitute A_0 value in equation (6)

$$a_n = (A + Bn)(-2)^n + 8/9 \quad \text{---} (7)$$

Initial conditions $a_0 = 1$, $a_1 = 2$

if $n = 0$.

Substitute $n = 0$ in eq (7).

$$a_0 = (A + B \times 0)(-2)^0 + 8/9$$

$$1 = A + \frac{8}{9}$$

$$A = 1 - \frac{8}{9} = \frac{9-8}{9} = 1/9$$

if $n = 1$.

Substitute $n = 1$ in eq (7)

$$a_1 = (A + B \times 1)(-2)^1 + 8/9$$

$$2 = \cancel{-2A} - 2B + 8/9$$

$$-2A - 2B = 2 - 8/9.$$

$$\cancel{-2B = 2A + 2 - \frac{8}{9}}$$

$$\cancel{= \frac{2}{9} - \frac{8}{9} + 2 = \frac{-5}{9} + 2}$$

$$\cancel{-2B = \frac{-5+18}{9} = \frac{13}{9}}$$

$$-2A - 2B = \frac{18-8}{9} = \frac{10}{9}$$

$$-2B = \frac{10}{9} + 2A = \frac{10}{9} + 2\left(\frac{1}{9}\right).$$

$$= \frac{10+2}{9} = \frac{12}{9}.$$

$$B = \frac{-12}{18} = \frac{-6}{9} = \frac{-2}{3}.$$

$$\boxed{B = \frac{-2}{3}}$$

A & B values substituted in eq. (2).

$$\boxed{a_n = \left(\frac{1}{9} + \left(\frac{-2}{3}\right) n\right)(-2)^n + \left(\frac{8}{9}\right)}$$

Ex:- (Problem).

$$a_n = 3a_{n-1} + 2n \quad \text{with} \quad a_1 = 3.$$

$$a_n = 3a_{n-1} + 2^n.$$

Ex: Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$ with initial condition $a_0 = 1$.

Hence

Sol: Given Inhomogeneous solution is.

$$a_n - 3a_{n-1} = 2^n \quad \text{--- (1)}$$

The general solution for the given RR is

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

(i) The associated homogeneous equation is

$$a_n - 3a_{n-1} = 0 \quad \text{--- (2)}.$$

Characteristic equation of (2) is

$$\cancel{r^2} - \cancel{3r} = 0. \quad r - 3 = 0$$

$$\Rightarrow r = 3.$$

The homogeneous equation is.

$$\boxed{a_n^{(h)} = c_1 3^n} \quad \text{--- (3)}$$

(ii) The R.H.S of the recurrence relation is 2^n and 2 is not the characteristic root. let the particular solution of the recurrence relation be.

$$\boxed{a_n = A \cdot 2^n} \quad \text{--- (4)}$$

Using this equation in the given recurrence relation in (1).

we get.

$$A 2^n - 3 \cdot A 2^{n-1} = 2^n.$$

$$A 2^n - \frac{3A \times 2^n}{2} = 2^n$$

$$\cancel{2^n}\left(A - \frac{3}{2}A\right) = \cancel{2^n}$$

$$A - \frac{3}{2}A = 1.$$

$$\frac{2A - 3A}{2} = 1$$

$$\frac{-A}{2} = 1 \Rightarrow \boxed{A = -2}$$

Hence the particular equation is,

$$a_n^{(P)} = (-2)(2)^n = -2^{n+1}$$

Hence the general solution is:

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$\Rightarrow \boxed{a_n = c_1 3^n - 2^{n+1}}$$

Using the conditions:

$a_0 = 1$, Here $n=0$, we get.

$$a_0 = c_1 3^0 - 2^{0+1}$$

$$\cancel{a_n = 3c_1 - 2^{0+1}} \Rightarrow \cancel{1 = 3c_1 - 2}$$

$$\cancel{1 = 3c_1 - 2}$$

$$\cancel{3c_1 = 1 + 2} \Rightarrow \cancel{c_1 = 3/3 = 1}$$

$$a_0 = c_1 3^0 - 2^{0+1}$$

$$1 = c_1 - 2$$

$$\boxed{c_1 = 3}$$

The required solution is $a_n = 3(3^n) - 2^{n+1}$

i.e $\boxed{a_n = 3^{n+1} - 2^{n+1}}$

This is required solution.

Ex: Solve the Recurrence relation

$$a_n = 2a_{n-1} + 2^n, \quad a_0 = 2.$$

Sol: The given recurrence relation is:

$$\boxed{a_n - 2a_{n-1} = 2^n} \quad \text{——— (1)}$$

(i) The associated homogeneous equation is

$$a_n - 2a_{n-1} = 0 \quad \text{——— (2)}$$

The characteristic equation is

$$r - 2 = 0$$

$$\boxed{r = 2}$$

The homogeneous solution is:

$$\boxed{a_n = c_1 2^n} \quad \text{——— (3)}$$

(ii) Since the R.H.S of the recurrence relation is 2^n and 2 is the characteristic root. Let

$a_n = n \times A 2^n$. — be a particular solution.

of the Recurrence relation.

Using this equation in the given relation, we get.

$$n \times A 2^n - 2A(n-1) \cdot 2^{n-1} = 2^n,$$

$$\cancel{An2^n - 2An\,2^{n-1} + 2A\,2^{n-1} = 2^n},$$

$$\cancel{An2^n - 2A(n}$$

$$nA2^n - A(n-1)2^{n-1+1} = 2^n.$$

$$nA2^n - A(n-1)2^n = 2^n.$$

$$A\cancel{2^n}(n - (n-1)) = \cancel{2^n}$$

$$n(\cancel{n} - n + 1) = 1 \qquad \boxed{A = 1}$$

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$$\boxed{a_n^{(p)} = n 2^n.}$$

Hence the general solution is.

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

$$\Rightarrow \boxed{a_n = c_1 2^n + n 2^n}$$

Given that $a_0 = 2$.

$$a_0 = c_1 2^0 + 0 \cdot 2^0$$

$$2 = c_1 \Rightarrow \boxed{c_1 = 2}$$

There fore the required solution is.

$$a_n = 2 \cdot (2^n) + n(2^n).$$

$$\boxed{a_n = 2^{n+1} + n \cdot 2^n.}$$

Ex: Solve the following recurrence relation.

$$a_{n+1} - 2a_n = 2^n, \quad n \geq 0 \text{ with } a_0 = 1.$$

Sol: Given recurrence relation

$$a_{n+1} - 2a_n = 2^n \quad \text{—— (1)}$$

The given RR is first order non-linear (or) non homogeneous recurrence relations.

The General Solution is,

$$a_n = a_n^{(h)} + a_n^{(p)} \quad \text{—— (2)}$$

$3n^2$.

$= 3n^2$

$= 3.$

To obtain $a_n^{(h)}$, put $f(n)=0$ in eq ①.

$$a_{n+1} - 2a_n = 0.$$

$$a_{n+1} = 2a_n \quad ③.$$

The general solution of homogeneous first order recurrence relation is.

$$a_n = c^n \cdot a_0 \quad ④$$

If $a_n = c^n$ then

$$a_{n+1} = c^{n+1}$$

$$2 \cdot a_n = c^{n+1}$$

$$2 \cdot \cancel{c^n} \cdot a_n = \cancel{c^n} \cdot c$$

$$2 a_n = c. \quad \boxed{\therefore c = 2}$$

c value can be substituted in equation ④.

$$a_n = 2^n \cdot a_0$$

$$\boxed{\therefore a_n^{(h)} = 2^n \cdot a_0}$$

Step 2: To obtain $a_n^{(p)}$, put $f(n) = 2^n$.

$$a_{n+1} - 2a_n = 2^n.$$

It is in the form of $f(n) = (b)^n$

$f(n) =$

Example:

1. $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$.

2. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 6$.

3. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2, a_1 = 5, a_2 = 15$.

4. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$.

$a_0 = 1, a_1 = -2, a_2 = -1$.

5. $a_n = 3a_{n-1} + 2n$, $a_1 = 3$. (root: $r_1 = 3$).

6. $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ (root: $r_1 = 3, r_2 = 2$)

when = $F(n) = 3^n$

$F(n) = n3^n$

$F(n) = n^2 2^n$

$r_1 = 3$, multiplicity 2.

$F(n) = (n^2+1)3^n \rightarrow (d_0 + d_1 n + d_2 n^2) =$

7. $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$.

8. Solve $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)^2$. $a_0 = 0, a_1 = 1$

Ex: $F(n) = (n+1) \cdot 2^n \rightarrow n^2 d \cdot 2^n$.

$\downarrow (d_1 + d_2 n)$

$F(n) = 3n + 2^n = a_n^{(P_1)} + a_n^{(P_2)}$.

$\downarrow (d_0 + d_1 n)$ $\quad d \cdot 2^n$

Ex: Solve the RR $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$.
where $a_0 = 1$, $a_1 = 1$.

Sol:- Given Recurrence Relation is.

$$a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n.$$

$$a_n - 4a_{n-1} + 4a_{n-2} = 3n + 2^n \quad \text{—} (1)$$

Step1: Associated Homogeneous Recurrence Relation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 \qquad (\because f(n) = 0).$$

$$r^2 - 4r + 4 = 0.$$

$$(r-2)^2 = 0 \qquad r - 2 = 0$$
$$r - 2 = 0$$

$$\therefore r = 2, 2.$$

roots are equal & real so. General homogeneous recurrence relation is.

$$a_n^{(h)} = (d_0 + d_1 n)\, 2^n. \,/\, (c_0 + c_1 n)\, 2^n.$$

Step2: R.H.S $\Rightarrow 3n + 2^n$.

particular solution: $a_n^{(P_1)} + a_n^{(P_2)}$.

Since the part of the R.H.S is $3n$ i.e a linear polynomial let.

$$a_n^{(P_1)} = d_0 + d_1 n. \quad \text{—} (2).$$

Substitute (2) in eq. (1)

$$(d_0 + d_1 n) - 4(d_0 + d_1(n-1)) + 4(d_0 + d_1(n-2)) = 3n$$

$$d_0 + d_1 n - \cancel{4d_0} - \cancel{4d_1 n} + 4d_1 + \cancel{4d_0} + \cancel{4d_1 n} - 8d_1 = 3n$$

$$\cancel{d_0 + 8d_1 + 5d_1 = 3n}. \qquad d_0 + d_1 n - 4d_1 = 3n.$$

$$d_0 + \quad (d_0 - 4d_1) + d_1 n = 3n.$$

Equating the co-efficients of n both sides we get:

$$d_1 = 3$$

$$d_0 - 4d_1 = 0$$

$$d_0 - 4 * 3 = 0$$

$$d_0 - 12 = 0 \Rightarrow d_0 = 12.$$

$\therefore$ Therefore, particular solution corresponding to $3n$ is:

$$\boxed{a_n^{(p_1)} = 12 + 3n.}$$

next, part of the R.H.S is 2^n and 2 is the double root of the

$$a_n^{(p_2)} = d \cdot 2^n \cdot n^2 \quad \text{--- (3)}$$

Substitute (3) in eq. (1)

$$\cancel{d \cdot n^2 2^n - 4d 2^{n-1}(n-1)^2 + 4d 2^{n-2}(n-2)^2 = d \cdot 2^n}$$

$$\cancel{d \cdot n^2 2^n - [4 \cdot 2^n (n^2+1-2n)] + [4 2^n (n^2+4-4n)] = 2^n}$$

$$\cancel{d \cdot n^2 2^n - 4n^2 2^n - 4 \cdot 2^n + 8n 2^n + 4n^2 2^n + 16 2^n - 16n 2^n}$$

$$\cancel{d \cdot n^2 2^n + 12 \cdot 2^n - 8n 2^n = 2^n} = 2^n.$$

$$d \cdot n^2 2^n - 4d \cdot 2^{n-1}(n-1)^2 + 4d \cdot 2^{n-2}(n-2)^2 = 2^n.$$

$$d \cdot n^2 \cdot 2^n - \frac{4d}{2} \cdot (n^2+1-2n) \cdot 2^n + \frac{4d}{4} \cdot (n^2+4-4n) \cdot 2^n = 2^n.$$

$$d \cdot n^2 \cdot 2^n - 2^n(2dn^2 + 2d - 4nd) + 2^n(dn^2 + 4d - 16nd) = 2^n.$$

$$2^n(\cancel{dn^2} - 2dn^2 - 2d + 4nd + \cancel{dn^2} + 4d - 16nd) = 2^n.$$

$$2d - 12nd = 1$$

$n = 0$

$$2d = 1 \Rightarrow d = 1/2.$$

$nd_2 + \ldots$

$2 = 3n^2.$

$d_2 = 3n^2.$

$d_2) = 3.$

$$a_n^{(P_2)} = \frac{1}{2} n^2 2^n$$

$$= n^2 2^{n-1}$$

$\therefore$ The particular solution is

$$a_n^{(P)} = a_n^{(P_1)} + a_n^{(P_2)}$$

$$\boxed{= 12 + 3n + n^2 2^{n-1}}$$

Hence the general solution is,

$$\boxed{a_n = (\alpha_0 + \alpha_1 n) \cdot 2^n + 3n + n^2 2^{n-1}}$$

Solve the recurrence relation

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2.$$

Sol: Given Recurrence relation is

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 \longrightarrow (1)$$

Step 1:- Associated homogeneous recurrence relation.

$$a_n + 5a_{n-1} + 6a_{n-2} = 0 \longrightarrow (2)$$

$$r^2 + 5r + 6 = 0$$

$$r(r+2) + 3(r+2) = 0.$$

$$r+2=0 \qquad r+3=0$$

$$r = -2, -3.$$

roots are real & distinct. so general. homogeneous solution is,

$$\boxed{a_n^{(h)} = d_1(-2)^n + d_2(-3)^n.}$$

Step 2:- find the particular solution. R.H.S of given RR is. $3 \cdot n^2$.

$$\Rightarrow (d_0 + d_1 n + d_2 n^2) \longrightarrow (3)$$

substitute eq (3) in eq. (1).

$$(d_0 + d_1 n + d_2 n^2) + 5(d_0 + d_1(n-1) + d_2(n-1)^2) + 6(d_0 + d_1(n-2) + d_2(n-2)^2) = 3n^2.$$

$$d_0 + d_1 n + d_2 n^2 + 5d_0 + 5nd_1 - 5d_1 + 5d_2 n^2 + 5d_2 - 10nd_2 + 6d_0 + 6nd_1 - 12d_1 + 6d_2 n^2 + 4d_2 - 4nd_2 = 3n^2.$$

$$12d_2 n^2 + 12d_0 - 17d_1 + nd_1 - 14nd_2 + 4d_2 = 3n^2.$$

$$12d_2 n^2 + 12d_0 - 17d_1 + 4d_2 + n(d_1 - 14d_2) = 3n^2$$

$12d_2 = 3$

$d_2 = 3/12 = 1/3.$

$d_1 - 14d_2 = 0.$

$d_1 - \frac{14}{3} = 0 \Rightarrow d_1 = \frac{14}{3}.$

$12d_0 - 17d_1 + 4d_2 = 0.$

$12d_0 - 17\cdot\frac{14}{3} + 4\cdot\frac{1}{3} = 0.$

$12d_0 - \frac{268}{3} = \frac{4}{3}$

$12d_0 = \frac{4}{3} + \frac{268}{3}$

$= \frac{272}{3} \Rightarrow d_0 = \frac{272}{3} - 12.$

$$\begin{array}{r} 17 \\ 14 \\ \hline 98 \\ 17 \\ + \\ \hline 268 \end{array}$$

$a_n^{(P)}$: R.H.S. constant $= d$.

$d + 5d + 6d = 3.$

$12d = 3 \Rightarrow d = 3/12 = 1/4.$

Another one n^2

$\boxed{d_0 + d_1 n + d_2 n^2}$

$(d_0 + d_1 n + d_2 n^2) + 5(d_0 + d_1(n-1) + d_2(n-1)^2) +$

$6(d_0 + d_1(n-2) + d_2(n-2)^2) = n^2.$

$d_0 + d_1 n + d_2 n^2 + 5(d_0 + d_1 n - d_1 + n^2 d_2 + 4d_2 - 4nd_2).$

$6(d_0 + d_1 n - 2d_1 + d_2 n^2 + 4d_2 - 4nd_2) = n^2$

$d_0 + d_1 x + d_2 x^2 + 5d_0 + 5d_1 x - 5d_1 + 5x^2 d_2$

$+ 20d_2 - 20x d_2 + 6d_0 + 6d_1 x - 12d_1$

$+ 6d_2 x^2 + 24d_2 - 24x d_2 = x^2.$

$x^2 (d_2 + 5d_2 + 6d_2)$

$d_2 - 13d_2 = 1$

$d_2 = 1/13.$

per year, with the interest compounded annually. How much money will be in the account after 30 years?

10.8. A factory makes custom sports vehicles at an increasing rate. In the first month only one vehicle is made, in the second month two vehicles are made an so on, with n vehicles made in the n^{th} month:

(a) Set up a recurrence relation for the number of vehicles produced in the first n months by this factory.

(b) How many vehicles are produced in the first year?

(c) Find an explicit formula for the number of vehicles produced in the first n months by this factory.

10.1.2 Solution of Linear Homogeneous Recurrence Relations with Constant Coefficients

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \tag{10.4}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$, is called a linear homogeneous recurrence relation of degree k with constant coefficients.

Note 10.1 The recurrence relation given in Eq. (10.4) is linear, since each a_i has the power 1 and no terms of the type $a_i a_j$ occurred.

Note 10.2 The *degree* of the recurrence relation is k, since a_n is expressed in terms of the previous k terms of the sequence, i.e., degree is the difference between the greatest and lowest subscripts of the members of the sequence occurring in the recurrence relation.

Note 10.3 The coefficients of the terms of the sequence are all *constants*. They are not functions of n.

Note 10.4 If $F(n) = 0$, then the recurrence relation is said to be *homogeneous*; otherwise, it is said to be *non-homogeneous*.

The recurrence relation given in Eq. (10.4) is homogeneous.

Example 10.9 Provide some examples of linear homogeneous recurrence relation. Also, give their degrees.

Solution (i) The recurrence relation $S_n = (0.09)S_{n-1}$ is a linear homogeneous recurrence relation of degree 1.

(ii) The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree 2.

(iii) The recurrence relation $a_n = a_{n-4}$ is a linear homogeneous recurrence relation of degree 4. ■

Example 10.10 Determine whether the following recurrence relations are linear homogeneous recurrence relations with constant coefficients:

(i) $a_n = 2a_{n-4} + a_{n-3}^2$

(ii) $H_n = 2H_{n-1} + 2$

(iii) $B_n = nB_{n-1}$

Solution (i) The recurrence relation $a_n = 2a_{n-4} + a_{n-3}^2$ is not linear.

(ii) The recurrence relation $H_n = 2H_{n-1} + 2$ is not homogeneous.

(iii) The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

Example 10.11 Determine which of the following recurrence relations are linear homogeneous recurrence relations with constant coefficients and also find their degrees:

(i) $a_n = 3a_{n-1} + 4a^2_{n-2} + 5a_{n-3}$

(ii) $a_n = 2na_{n-1} + a_{n-2}$

(iii) $a_n = a_{n-1} + a_{n-4}$

(iv) $a_n = a_{n-1} + 2$

(v) $a_n = a_{n-1}^2 + a_{n-2}$

(vi) $a_n = a_{n-2}$

(vii) $a_n = a_{n-1} + n$

Solution (i) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = (n-3) - n = 3.$$

(ii) This does not have constant coefficients.

(iii) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = (n-4) - n = 4.$$

(iv) This is not a homogeneous recurrence relation.

(v) This is not a linear recurrence relation.

(vi) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = 2.$$

(vii) This is not a homogeneous recurrence relation.

Characteristic roots Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

The characteristic equation of the recurrence relation given above is

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k \neq 0$$

The solutions of the characteristic equation are called the *characteristic roots*.

Theorem 10.1 Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof Let r_1 and r_2 be two distinct roots of the characteristic equation $r^2 - c_1 r - c_2 = 0$.

Let α_1, α_2 be two constants such that $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$. (10.5)

We need to prove that $\{a_n\}$ is a solution of the recurrence relation.

Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, we have

$$r_1^2 - c_1 r_1 - c_2 = 0 \Rightarrow r_1^2 = c_1 r_1 + c_2 \qquad (10.6)$$

and $r_2^2 - c_1 r_2 - c_2 = 0 \Rightarrow r_2^2 = c_1 r_2 + c_2$ (10.7)

Now, $c_1 a_{n-1} + c_2 a_{n-2}$

$$= c_1 [\alpha_1 r_1^{\,n-1} + \alpha_2 r_2^{\,n-1}] + c_2 [\alpha_1 r_1^{\,n-2} + \alpha_2 r_2^{\,n-2}]$$

$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)\,.$$

$$= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \quad \text{[from Eqs (10.6) and (10.7)]}$$

$$= \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$= a_n \quad \text{[by our assumption]}$$

$\therefore$ The sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

Conversely, we assume that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \text{ for some constants } \alpha_1, \alpha_2 \text{ and } n = 0, 1, 2, \ldots.$$

We need to prove that every solution $\{a_n\}$ of the recurrence relations has the form

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for some constants } \alpha_1 \text{ and } \alpha_2 \text{ and } n = 0, 1, 2, \ldots.$$

Suppose $\{a_n\}$ is a solution of the recurrence relation and the initial conditions $a_0 = c_0$ and $a_1 = c_1$ hold. We need to show that there are constants α_1 and α_2 so that the sequence $\{a_n\}$ with

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ satisfies the initial conditions.}$$

Now

$$a_0 = c_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 = \alpha_1 + \alpha_2$$

$$a_1 = c_1 = \alpha_1 r_1 + \alpha_2 r_2 \quad \text{[by our assumption]}$$

i.e., $c_0 = \alpha_1 + \alpha_2$

and $c_1 = \alpha_1 r_1 + \alpha_2 r_2$

Solving this, we get

$$\alpha_1 = \frac{c_1 - c_0 r_2}{r_1 - r_2}$$

and $\alpha_2 = \dfrac{c_0 r_1 - c_1}{r_1 - r_2}$

The values of α_1 and α_2 are valid only if $r_1 \neq r_2$.

Therefore, for the above values of α_1 and α_2, the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

Since $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with $a_0 = c_0$ and $a_1 = c_1$ uniquely determine the sequence, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution form.

Theorem 10.2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof First, we show that if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Since r_0 is a root of the characteristic equation $r^2 - c_1 r - c_2 = 0$

r_0 is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ [by Theorem 10.1] (10.8)

Now we need to prove that $a_n = n r_0^n$ is also a solution of Eq. (10.8).

Since r_0 is a root of $r^2 - c_1 r - c_2 = 0$ (10.9)

and the degree of equation (10.9) is 2

$$r^2 - c_1 r - c_2 = (r - r_0)^2$$
$$= r^2 - 2r_0 r + r_0^2$$

Equating the corresponding coefficients, we have

$$c_1 = 2r_0,\ c_2 = -r_0^2$$

Now

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1 [(n-1)r_0^{n-1}] + c_2 [(n-2)r_0^{n-2}]$$
$$= 2r_0 (n-1) r_0^{n-1} - r_0^2 (n-2) r_0^{n-2}$$
$$= r_0^n [2(n-1) - (n-2)]$$
$$= n r_0^n$$
$$= a_n$$

$\therefore\ n r_0^n$ is a solution.

Hence, by Theorem 10.7, $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ is a solution of Eq. (10.8).

Conversely, we have to prove that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for some constants α_1 and α_2 and for $n = 0, 1, 2, \ldots$.

Suppose that $\{a_n\}$ is a solution of the recurrence relation and the initial conditions $a_0 = c_0$ and $a_1 = c_1$ hold. We need to show that there are constants α_1 and α_2 so that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ satisfies the initial conditions.

$$a_0 = c_0 = \alpha_1$$
$$a_1 = c_1 = \alpha_1 r_0 + \alpha_2 r_0$$

i.e., $c_1 - \alpha_1 r_0 = \alpha_2 r_0$

i.e., $\alpha_2 = \left[\dfrac{c_1 - c_0 r_0}{r_0}\right]$

Therefore, when $\alpha_1 = c_0$, $\alpha_2 = \left[\dfrac{c_1 - c_0 r_0}{r_0}\right]$, the sequence $\{a_n\}$ with $\alpha_1 r_0^n + \alpha_2 n r_0^n$ satisfies the two initial conditions.

Since the recurrence relation and these initial conditions uniquely determine the sequence,

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

Theorem 10.3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$ where $a_1, a_2, \ldots, a_k$ are constants.

Theorem 10.4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ *for* $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1})$$
$$r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{mt-1}) r_t^n,$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \le i \le t$.

Note 10.5 Consider the recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $n \ge 2$.

The characteristic equation is $r^2 - c_1 r - c_2 = 0$
Let the roots of the characteristic equation be r_1 and r_2.

Case (i) If r_1 and r_2 are real and distinct, then the solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are arbitrary constants.

Case (ii) If r_1 and r_2 are real and equal, then the solution is $a_n = (\alpha_1 + \alpha_2 n) r^n$ where α_1 and α_2 are arbitrary constants.

Case (iii) If r_1 and r_2 are complex numbers, then the solution is $a_n = r^n (\alpha_1 \cos n\theta + \alpha_2 \cos n\theta$, where α_1 and α_2 are arbitrary constants.

Example 10.12 Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, \text{ for } n \ge 2,\ a_0 = 1,\ a_1 = 0.$$

Solution The given recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$.
The characteristic equation of the recurrence relation is $r^2 - 5r + 6 = 0$.

$\therefore r = 2, 3$

Hence, the solution is $a_n = c_1 2^n + c_2 3^n$, where c_1 and c_2 are constants. (10.10)

Initial conditions are $a_0 = 1, a_1 = 0$.

Now, $a_0 = 1 \Rightarrow c_1 2^0 + c_2 3^0 = 1$

$\Rightarrow c_1 + c_2 = 1$ (10.11)

$$a_0 = 0 \Rightarrow c_1 2^1 + c_2 3^1 = 0$$

$\Rightarrow 2c_1 + 3c_2 = 0$ (10.12)

Solving Eqs (10.11) and (10.12) we have, respectively,

$\Rightarrow c_1 = 1 - c_2$

$\Rightarrow 2(1 - c_2) + 3c_2 = 0$

$\Rightarrow c_2 = -2$

Also, $c_1 = 1 - (-2) = 3$

$\therefore c_1 = 3$ and $c_2 = -2$

Hence, the unique solution is

$$a_n = 3(2^n) - 2(3^n) \text{ for } n \ge 2$$

Example 10.13 Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2} \text{ for } n \ge 2,\ a_0 = 16,\ a_1 = 80.$$

Solution The given recurrence relation is

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 8r + 16 = 0$$

$\Rightarrow (r-4)^2 = 0$

$\Rightarrow r = 4, 4$

Hence, the solution is

$$a_n = c_1 4^n + c_2 n 4^n, \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.} \quad (10.13)$$

Initial conditions:

$$a_0 = 16 \Rightarrow c_1.4^0 + c_2.0 = 16$$

$\Rightarrow c_1 = 16$

Also, $a_0 = 80 \Rightarrow c_1 4 + c_2 1\,(4^1) = 80$

$\Rightarrow 4c_1 + 4c_2 = 80$

$\Rightarrow c_1 + c_2 = 20$

$\Rightarrow c_2 = 4$, since $c_1 = 16$

Hence, the unique solution is

$$a_n = 16\,(4^n) + 4n\,(4^n) = 4^{n+2} - n4^{n+1}$$

$\Rightarrow a_n = (4-n)\,4^{n+1}, n \geq 2$

Example 10.14 Find the solution of the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic equation is

$$r^3 - 2r^2 - r + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow (r-1)\,(r^2 - r - 2) = 0$

$\Rightarrow (r-1)\,(r+1)\,(r-2) = 0$

$\Rightarrow r = 1, 2, -1$

Hence, the solution

$$a_n = c_1 1^n + c_2 2^n + c_3(-1)^n, \text{ where } c_1, c_2, c_3 \text{ are arbitrary constants.} \quad (10.14)$$

Initial conditions are $a_0 = 3$ $a_1 = 6$ and $a_2 = 0$.

When $a_0 = 3$, $c_1 + c_2 + c_3 = 3$ (10.15)

When $a_1 = 6$, $c_1 + c_2 2^1 + c_3(-1)^1 = 6$

$\Rightarrow c_2 + 2c_2 - c_3 = 6$ (10.16)

When $a_2 = 0$, $c_1 + c_2 2^2 + c_3(-1)^2 = 0$

$\Rightarrow c_2 + 4c_2 + c_3 = 0$ (10.17)

Adding Eqs (10.15) and (10.16) $\Rightarrow 2c_1 + 3c_2 = 9$ (10.18)

Adding Eqs (10.16) and (10.17) $\Rightarrow 2c_1 + 6c_2 = 6$ (10.19)

Subtracting Eq. (10.19) from Eq. (10.20) $\Rightarrow -3c_2 = 3$

$\Rightarrow c_2 = -1$

Equation (10.18) $\Rightarrow 2c_1 = 9 - 3c_2$

$\Rightarrow 2c_1 = 9 + 3$

$\Rightarrow c_1 = 6$

Equation (10.15) $\Rightarrow c_3 = 3 - c_1 - c_2$

$= 3 - 6 + 1$

$\Rightarrow c_3 = -2$

$\therefore$ The unique solution is $a_n = 6(1^n) - 2^n - 2(-1)^n$.

Example 10.15 Find an explicit formula for the Fibonacci numbers.

Solution The Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ... The recurrence relation corresponding to the Fibonacci sequence $\{F_n\}$, $n \geq 0$, is

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial conditions $F_0 = 0$, $F_1 = 1$.

The characteristic equation of the recurrence relation is $r^2 - r - 1 = 0$.

Solving it, we have $r = \dfrac{1 \pm \sqrt{5}}{2}$

Hence, the solution of the recurrence relation is

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{10.20}$$

where c_1 and c_2 are arbitrary constants.

Initial conditions are $F_0 = 0$ and $F_1 = 1$.

Now, $F_0 = 0 \Rightarrow c_1 + c_2 = 0$ (10.21)

And $F_1 = 1 \Rightarrow c_1\left(\dfrac{1+\sqrt{5}}{2}\right) + c_2\left(\dfrac{1-\sqrt{5}}{2}\right) = 1$ (10.22)

Solving Eqs (10.21) and (10.22):

Equation (10.21) $\Rightarrow c_1 = -c_2$.

$\therefore$ Equation (10.22) $\Rightarrow c_2\left[\left(\dfrac{1-\sqrt{5}}{2}\right) - \left(\dfrac{1+\sqrt{5}}{2}\right)\right] = 1$

$\Rightarrow c_2\left[-\sqrt{5}\right] = 1$

$\Rightarrow c_2 = \dfrac{-1}{\sqrt{5}}$

Using this in Eq. (10.21), we get

$$c_1 = \frac{1}{\sqrt{5}}$$

The solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 0$$

Example 10.16 The *Lucas numbers* satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ and the initial conditions $L_0 = 2$ and $L_1 = 1$.

(i) Show that $L_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, \ldots$, where F_n is the nth Fibonacci number.

(ii) Find an explicit formula for the Lucas numbers.

Solution (i) Let S_n be the statement

$$L_n = F_{n-1} + F_{n+1} \text{ for } n = 2, 3, \ldots.$$

We shall prove this by using the principle of mathematical induction.

Basic step: S_2 is shown to be true. That is, we need to prove

$$L_2 = F_1 + F_3.$$

By the definition of Lucas number, we can write

$$\begin{aligned} L_2 &= L_1 + L_0 \\ &= 1 + 2, \text{ since } L_0 = 2, L_1 = 1 \\ &= 3 \end{aligned}$$

R.H.S. $S_2 = F_1 + F_3$

$= 1 + 2$, since from the Fibonacci number $0(F_0)\ 1(F_1)\ 1(F_2)\ 2(F_3)\ 3(F_4) \ldots$

$= 3$

$\Rightarrow L_2 = F_1 + F_3$

Hence, S_2 is true.

Inductive step: We assume that S_k is true for every $k \leq n$.

$\Rightarrow L_k = F_{k-1} + F_{k+1}$ is true for every $k = n$.

We need to prove that S_{k+1} is true. That is, we need to prove

$L_{k+1} = F_k + F_{k+2}$.

Now, by the definition of Lucas numbers

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} \\ &= (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k) \quad \text{[by our assumption]} \\ &= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k) \\ &= F_k + F_{k+2} \quad \text{[by the definition of Fibonacci numbers]} \end{aligned}$$

i.e, S_{k+1} is true.

Hence, by the principle of mathematical induction,

S_n is true for every n.

(ii) The given recurrence relation is

$L_n - L_{n-1} - L_{n-2} = 0$

The corresponding characteristic equation is

$$r^2 - r - 1 = 0$$

$$\Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

Let $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$

Therefore, the solution is

$$L_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad (10.23)$$

where c_1 and c_2 are constants.

Initial conditions are $L_0 = 2$, $L_1 = 1$.

Now, $L_0 = 2 \Rightarrow c_1 + c_2 = 2$ (10.24)

$$L_1 = 1 \Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad (10.25)$$

Solving Eqs (10.24) and (10.25), we get

$$\alpha_1 = 1 \text{ and } \alpha_2 = 1$$

Hence, the unique solution is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example 10.17 Solve the recurrence relation

$$a_n = 2a_{n-1} - 2a_{n-2},\ a_0 = 1,\ a_1 = 2.$$

Solution The given recurrence relation is $a_n - 2a_{n-1} + 2a_{n-2} = 0$.

Its characteristic equation is $r^2 - 2r + 2 = 0$

$$\Rightarrow r = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

The modulus–amplitude form of

$$1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}\right)$$

$\therefore$ The general solution of the recurrence relation is

$$a_n = \left(\sqrt{2}\right)^n \left(c_1 \cos\frac{n\pi}{4} \pm c_2 \sin\frac{n\pi}{4}\right) \tag{10.26}$$

where c_1 and c_2 are constants.

Given that $a_0 = 1$ and $a_1 = 2$.

Now, $a_0 = 1 \Rightarrow c_1 \pm 0 = 1$

$\Rightarrow c_1 = 1.$

Also, $a_1 = 2 \Rightarrow \left(\sqrt{2}\right)\left(c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4}\right) = 2$

$$\Rightarrow \sqrt{2}\left[c_1 \frac{1}{\sqrt{2}} + c_2 \frac{1}{\sqrt{2}}\right] = 2$$

$\Rightarrow c_1 + c_2 = 2$

$\Rightarrow c_2 = 1.$

$\therefore$ The required solution is

$$a_n = \left(\sqrt{2}\right)\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$$

■

10.1.3 Solution of Non-homogeneous Recurrence Relation or Inhomogeneous Recurrence Relation

A linear inhomogeneous or non-homogeneous recurrence relation with constant coefficients of degree k is a recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + G(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $G(n)$ is a function not identically zero depending only on n.

Algorithm for solving non-homogeneous finite-order linear recurrence relation To solve the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = G(n)$$

or

$$S(k) = c_1 S(k-1) + c_2 S(k-2) + \cdots + c_n S(k-n) = g(k),$$

we have to adopt the following procedure.

Step 1. We obtain the homogeneous solution.

First, we write the associated homogeneous recurrence relation, namely $G(n) = 0$

i.e., $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_n a_{n-k} = 0$

Then, we find its general solution, which is called the homogeneous solution.

Step 2. We obtain the particular solution.

There is no general procedure for finding the particular solution of a recurrence relation.

However, if $G(n)$ has any one of the following forms

(i) polynomial in n
(ii) a constant
(iii) powers of constant

then we may guess the forms of the particular solution and exactly find out by the method of undetermined coefficients.

Particular solution for given G(n) Table 10.2 shows the particular solution for given $G(n)$.

Table 10.2 Particular solution for given $G(n)$

S. No.	G(n)	*Form of particular solution*
(i)	A constant, c	A constant, d
(ii)	A linear function $c_0 + c_1 n$	A linear function $d_0 + d_1 k$
	N	$d_0 + d_1 k$
(iii)	n^2	$d_0 + d_1 k + d_2 k^2$
(iv)	An m[th] degree polynomial	An m^{th} degree polynomial
	$c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$	$d_0 + d_1 k + d_2 k^2 + \cdots + d_m k^m$
(v)	$r^n, r \in R$	dr^n

Step 3. We substitute the guess from Step 2 into the recurrence relation. If the guess is correct, then we can determine the unknown coefficient of the guess. If we are not able to determine the constants, then our guess is wrong and hence we go to Step 2.

Step 4. The general solution of the recurrence relation is the sum of the homogeneous and particular solutions.

Step 10. If no initial conditions are given, then Step 4 will give the solution.

If n initial conditions are given, then we get n equations with n unknowns. Solving the system, we get a complete solution.

Example 10.18 Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$

Solution The inhomogeneous recurrence relation is

$$a_n - 3a_{n-1} = 2^n \qquad (10.27)$$

(i) The associated homogeneous equation is

$$a_n - 3a_{n-1} = 0$$

$r^1 - 3 = 0$

$a_n - 3a_{n-1} + 4a_{n-2}$

$r^2 - 3 \cdot r^1 + 4 = 0$

Its characteristic equation is

$$r - 3 = 0$$

$\Rightarrow r = 3$

$r = 3, 3$

$\therefore$ The homogeneous solution is $a_1 r^n$ $n c_1 r^n$

$$a_n(H) = c_1 3^n$$

$c_1 3^n + c_2 n 3^n$

(ii) Since the R.H.S. of the recurrence relation is 2^n and 2 is not the characteristic root, let the particular solution of the recurrence relation be

$$a_n = d2^n$$

Using this equation in the given recurrence relation, we get

$$d2^n - 3d2^{n-1} = 2^n$$

$\Rightarrow d - \frac{3}{2}d = 1$

$\Rightarrow 2d - 3d = 2$

$\Rightarrow d = -2$

$\therefore\ a_n^{(P)} = -2\,(2)^n = -2^{n+1}$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$\Rightarrow a_n = c_1 3^n - 2^{n+1}$ (10.28)

Using the condition $a_0 = 1$, we get

$$a_0 = c_1 3^0 - 2^1 = 1$$

$\Rightarrow c_1 - 2 = 1$

$\Rightarrow c_1 = 3$

$\therefore$ The required solution is $a_n = 3\,(3^n) - 2^{n+1}$

i.e., $a_n = 3^{n+1} - 2^{n+1}$

Example 10.19 Solve the recurrence relation $a_n = 2a_{n-1} + 2^n$, $a_0 = 2$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} = 2^n \tag{10.29}$$

(i) The associated homogeneous equation is $a_n - 2a_{n-1} = 0$.

The characteristic equation is

$$r - 2 = 0$$

$\Rightarrow r = 2$

$\therefore$ The homogeneous solution is $a_n^{(H)} = c_1 2^n$.

(ii) Since the R.H.S. of the recurrence relation is 2^n and 2 is the characteristic root, let $a_n = dn2^n$ be a particular solution of the recurrence relation.

Using this equation in the given recurrence relation, we get

$dn2^n - 2d(n-1)2^{n-1} = 2^n$

$\Rightarrow dn2^n - d(n-1)2^{n-1} = 2^n$

$\Rightarrow d\,[n - (n-1)] = 1$

$\Rightarrow d = 1$

$$a_n^{(P)} = n2^n$$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_N^{(P)}$$

$\Rightarrow a_n = c_1 2^n + n2^n$ (10.30)

Given that $a_0 = 2$,

$\therefore\ c_1 2^0 + 0 = 2$

$\Rightarrow c = 2$

Therefore, the required solution is

$$a_n = 2(2^n) + n2^n$$

$$\Rightarrow a_n = (2 + n) + n2^n$$

■

Example 10.20 Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2$, with $a_0 = 25$, $a_1 = 16$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} + a_{n-2} = 2 \tag{10.31}$$

(i) The associated homogeneous equation is $a_n - 2a_{n-1} + a_{n-2} = 0$

Its characteristic equation is $r^2 - 2r + 1 = 0$

$\Rightarrow (r - 1)^2 = 0$

$\Rightarrow r = 1, 1.$

$\therefore$ The homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)1^n$$

(ii) Since the R.H.S. of the recurrence relation is 2, a constant, we assume the particular solution of the recurrence to be

$$a_n^{(H)} = d, \text{ a constant.}$$

Using this solution in the given recurrence relation, we get

$$d - 2d + d = 2$$

i.e., $0 = 2$, which is impossible.

Thus, our assumption is wrong.

Now, we assume that $a_n^{(p)} = nd$

Using this solution in the given recurrence relation, we get

$$nd - 2(n-1)d + (n-2)d = 2$$

$$\Rightarrow nd - 2nd + nd + 2d - 2d = 2$$

$\Rightarrow 0 = 2$, which is also impossible.

Thus, our assumption is wrong.

Now, we assume that $a_n^{(H)} = n^2 d$

Using this solution in the given recurrence relation, we get

$$n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$$

$$\Rightarrow n^2 d - 2(n^2 - 2n + 1)d + (n^2 - 4n + 4)d = 2$$

$$\Rightarrow n^2 - 2n^2 d + 4nd - 2d + n^2 d - 4nd + 4d = 2$$

$$\Rightarrow d = 1$$

$$\therefore a_n^{(p)} = n^2$$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_n^{(p)}$$

$$\Rightarrow a_n = [c_1 + c_2 n]1^n + n^2 \tag{10.32}$$

Given $a_0 = 25$ and $a_1 = 16$.

Now, $a_0 = 25$

$\Rightarrow [c_1 + 0] + 0 = 25$

$\Rightarrow c_1 = 25$

Also, $a_1 = 16$

$\Rightarrow (c_1 + c_2) + 1^2 = 16$

$\Rightarrow c_1 + c_2 + 1 = 16$

$\Rightarrow c_2 = 16 - 1 - 25$

$= -10$

$\therefore$ The required solution is

$a_n = (25 - 10n)\, 1^n + n^2$

$\Rightarrow a_n = n^2 - 10n + 25$ ■

Example 10.21 Solve the recurrence relation

$$a_k - 7a_{k-1} + 10a_{k-2} = 6 + 8k,\ a_0 = 1,\ a_1 = 2$$

or solve

$$S(k) - 7S(k-1) + 10S(k-2) = 6 + 8k,\ S(0) = 1,\ S(1) = 2.$$

Solution The given recurrence relation is

$$a_k - 7a_{k-1} + 10a_{k-2} = 6 + 8k \qquad (10.33)$$

(i) The associated homogeneous equation is

$a_k - 7a_{k-1} + 10a_{k-2} = 0$

Its characteristic equation is

$$r^2 - 7r + 10 = 0$$

$\Rightarrow (r-5)(r-2) = 0$

$\Rightarrow r = 2, 5$

Therefore, the homogeneous solution is

$$a_k^{(H)} = c_1 2^k + c_2 5^k$$

(ii) We need to find the particular solution,

let $a_k^{(P)} = d_0 + d_1 k$, since the R.H.S. is a linear polynomial.

Using this solution in the given recurrence relation, we get

$$(d_0 + d_1 k) - 7[d_0 + d_1(k-1)] + 10[d_0 + d_1(k-2)] = 6 + 8k$$

$\Rightarrow (d_0 - 7d_0 + 10d_0) + d_1[k - 7(k-1) + 10(k-2)] = 6 + 8k$

$\Rightarrow 4d_0 + d_1[k - 7k + 7 + 10k - 20)] = 6 + 8k$

$\Rightarrow (4d_0 - 13d_1) + 4d_1 k = 6 + 8k$

Equating the corresponding coefficients on both sides, we get

$$4d_0 - 13d_1 = 6 \text{ and } 4d_1 = 8$$

Now, $4d_1 = 8 \Rightarrow d_1 = 2$

$\Rightarrow 4d_0 = 6 + 13\,(2)$

$\Rightarrow d_0 = 8$

$\Rightarrow a_k^{(P)} = 8 + 2k$

Thus, the general solution is $a_k = a_k^{(H)} = a_k^{(P)}$

$$\Rightarrow a_k = c_1 2^k + c_2 5^k + 8 + 2k \tag{10.34}$$

Given that, $a_0 = 1,\ a_1 = 2$

Now, $a_0\ 1 \Rightarrow\quad c_1 + c_2 + 8 = 1$

$$\Rightarrow c_1 + c_2 + 8 = -7 \tag{10.35}$$

Also, $a_1 = 2 \Rightarrow c_1 2 + c_2 5 + 8 + 2 = 2a$

$$\Rightarrow 2c_1 + 5c_2 = -8 \tag{10.36}$$

Solving Eqs (10.35) and (10.36), we get

$$c_1 = -9 \text{ and } c_2 = 2$$

∴ The required solution is

$$a_k = -9\,(2^k) + 2\,(5^k) + 8 + 2k$$

Example 10.22 Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)\,2^n$$

Solution The given recurrence relation is

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n \tag{10.37}$$

(i) The associated homogeneous equation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

Its characteristic equation is

$r^2 - 4r + 4 = 0$

i.e., $(r-2)^2 = 0$

$\Rightarrow r = 2, 2$

∴ The homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)2^n$$

(ii) Since the R.H.S. of the recurrence relation is $(n+1)2^n$ and 2, 2 is the characteristic root of the equation (i.e., 2 is repeated twice), we assume the particular solution of the recurrence relation to be

$$a_n^{(P)} = (c_1 + c_2 n)n^2 2^n$$

Using this solution in the given recurrence relation, we have

$$(c_1 + c_2 n)n^2 2^n - 4\,[c + c_2\,(n-1)]\,(n-1)^2\,2^{n-1} + 4\,[c_1 + c_2\,(n-2)]\,(n-2)^2\,2^{n-2} = (n+1)2^n$$

$$\Rightarrow 4(c_1 + c_2 n)n^2 - 8(n-1)^2[c_1 + c_2(n-1)] + 4(n-2)^2[c_1 + c_2(n-2)] = 4(n+1)$$

$$\Rightarrow 4(c_1 + c_2 n)n^2 - 8(n^2 - 2n + 1)[c_1 + c_2(n-1)] + 4(n^2 - 4n + 4)[c_1 + c_2(n-2)] = 4(n+1) \quad (10.38)$$

Putting, $n = 0$, we get

$$-8(c_1 - c_2) + 16(c_1 2c_2) = 4$$

$$\Rightarrow 8c_1 - 24c_2 = 4$$

$$\Rightarrow c_1 - 3c_2 = \frac{1}{2} \quad (10.39)$$

Equating the coefficients of n on both sides of Eq. (10.38), we get

$$16c_1 - 16c_2 - 8c_2 - 16c_1 + 32c_2 + 16c_2 = 4$$

$$\Rightarrow 24c_2 = 4$$

$$\Rightarrow c_2 = \frac{1}{6}$$

Therefore, Eq. (10.39) gives

$$c_1 = \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow c_1 = 1$$

$$\therefore\ a_n^{(P)} = \left(1 + \frac{1}{6}n\right)n^2 2^n$$

$$= \left(n^2 + \frac{n^3}{6}\right)2^n$$

Thus, the general solution of the recurrence relation is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$\Rightarrow a_n = \left[c_1 + c_2 n + n^2 + \frac{n^3}{6}\right]2^n$$

Example 10.23 Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$, $a_0 = 1$, $a_1 = 1$.

Solution The given recurrence relation is

$$a_n - 4a_{n-1} - 4a_{n-2} = 3n + 2^n \quad (10.40)$$

(i) The associated homogeneous equation is

$$a_n - 4a_{n-1} - 4a_{n-2} = 0$$

Its characteristic equation is $r^2 - 4r + 4 = 0$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r = 2, 2$$

Thus, the homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)2^n \tag{10.41}$$

(ii) R.H.S. = $3n + 2^n$

Particular solution = $a_n^{(P_1)} + a_n^{(P_2)}$

Since part of the R.H.S. is $3n$, i.e., a linear polynomial,

let $a_n^{(P_1)} = d_0 + d_1 n$

Using this solution in the recurrence relation, we get

$(d_0 + d_1 n) - 4\{d_0 + d_1(n-1)\} + 4\{d_0 + d_1(n-2)\} = 3n$

$\Rightarrow (d_0 - 4d_0 + 4d_0) + d_1[n - 4(n-1) + 4(n-2)] = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1 n = 3n$

Equating the coefficients of n on both sides, we get

$$d_1 = 3$$

Equating the constant terms on both sides, we get

$$d_0 - 4d_1 = 0$$

$\Rightarrow d_0 = 12$

Therefore, particular solution corresponding to $3n$ is

$$a_n^{(P_1)} = 12 + 3n \tag{10.42}$$

Since part of the R.H.S. is 2^n and 2 is the double root of the characteristic equation, let us assume the particular solution to be $a_n^{P_2} = dn^2 2^n$

Using this solution in the given recurrence relation, we get

$dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n$

$\Rightarrow 4dn^2 - 8d(n-1)^2 + 4d(n-2)^2 = 4$

$\Rightarrow dn^2 - 2d(n-1)^2 + d(n-2)^2 = 1$

$\Rightarrow dn^2 - 2d(n^2 - 2n + 1) + d(n^2 - 4n + 4) = 1$

Putting $n = 0$, we get

$-2d + 4d = 1$

$\Rightarrow d = 1/2$

Therefore, the particular solution corresponding to 2^n is

$$a_n^{(P_2)} = \frac{1}{2} n^2 2^n$$

$$= n^2 2^{n-1} \tag{10.43}$$

Therefore, the particular solution is

$$a_n^P = a_n^{(P_1)} + a_n^{(P_2)}$$

$$= 12 + 3n + n^2 2^{n-1}$$

Hence, the general solution is

$a_n (c_1 + c_2 n)2^n + 12 + 3n + n^2 2^{n-1}$

Given that $a_0 = 1, a_1 = 1$

$a_0 = 1 \Rightarrow c_1 + 12 = 1$

$\Rightarrow c_1 = -11$

Also, $a_1 = -1 \Rightarrow (c_1 + c_2)\,2 + 12 + 3 + 2^2 = 1$

$\Rightarrow 2c_1 + 2c_2 = -18$

$\Rightarrow c_1 + c_2 = -9$

$\Rightarrow c_2 = -9 - (-11) = 2$

$\Rightarrow c_2 = 2$

Thus, the required solution is

$$a_n = (2n - 11)\,2^n + 12 + 3n + n^2 2^{n+1}$$

Example 10.24 For what values of constants A and B is $a_k = Ak + B$ a solution of the recurrence relation $a_k = 2a_{k-1} + k + 10$.

Solution The given recurrence relation is $a_k = 2a_{k-1} + k + 10$. (10.44)

Now, $a_k = Ak + B$ is a solution of Eq. (10.44) if it satisfies the recurrence relation.

$\Rightarrow 2a_{k-1} + k + 5 = 2[A(k-1) + B] + k + 5$

$= (2A + 1)\,k + 2(B - A) + 5$

$= Ak + B$, when $A = (2A + 1)$ and $2(B - A) + 5 = B$

When $A = -1$ and $B = -7$, the above relation holds true

Thus, for $A = -1$ and $B = -7$, $a_k = Ak + B$ is one of the solutions.

EXERCISES

10.9 Determine which of these equations are linear homogeneous recurrence relations with constant coefficients and find their degree.

(a) $a_k = a_{k-1}^2$ (b) $a_k = \dfrac{a_k - 1}{k}$

(c) $a_k = a_{k-1} + a_{k-2} + k + 3$

(d) $a_k = 4a_{k-2} + 5a_{k-4} + 9a_{k-7}$

10.10 Solve the recurrence relation $a_n = \dfrac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

10.11 Solve the recurrence relation $a_n = 7a_{n-2} + 6a_{n-3}$, $a_0 = 9$, $a_1 = 10$, $a_2 = 32$.

10.12 Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$, with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.

10.13 Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 6$.

10.14 Solve the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$ and $a_2 = 110$.

10.15 Solve $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$, $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

10.16 Solve $a_n - a_{n-1} - 6a_{n-2} = -30$, with $a_0 = 20$ and $a_1 = -5$

10.17 Solve the recurrence relation $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$.

10.18 Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$, $n \geq 0$. $a_0 = 1$, $a_1 = 4$.

10.19 Solve the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$, with $a_0 = 1$, $a_1 = 4$.

10.20 What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?